Recitation 1

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Logistics

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Texts:	Peskin and Schroeder:	An Introduction to Quantum Field Theory	$(+,-,-,-)$
	Weinberg, Volume 1:	The Quantum Theory of Fields, Foundations	$(-,+,+,+)$
	Schwartz:	Quantum Field Theory and the Standard Model	$(+,-,-,-)$
	Srednicki:	Quantum Field Theory	$(-,+,+,+)$
			c.f. Hong: $(-,+,+,+)$

1 Preliminaries

1.1 Relativistic Quantum Mechanics: Why QFT?

Often, it is written that QFT = QM + SR. Physicists say this with nearing a century of experience bourne from confusion and pain due to the limitations of the theories they built to describe physical phenomena. When learning the subject, one is presented with a polished product that is often unmotivated, a big black box that works. Hence, as one is churning the heavy machinery that is QFT to produce some sensible result, it can be difficult to see why we need it at all. For instance, why do we need fields? But QFT is not abstraction for the sake of abstraction, and if there was a simpler theory that described particle physics we would have found it. In light of this, I'd like to begin today by making extremely explicit why quantum mechanics alone fails to describe the physics of very small scales.

Fact 1: Ordinary (non-relativistic) QM breaks down at short distances.
In ordinary QM, let us consider the simplest possible system–a particle in a box. By the uncertainty principle, we know that $\Delta x \Delta p \gtrsim \hbar$. Hence, if we confine our particle to scales $\sim \hbar/c$, then the momentum becomes relativistic, and we cannot ignore relativistic corrections.

Having realized this as a physicist in the mid-1920s, one's programme may be to extend quantum mechanics to include relativity. The naïve extension, which we can call 'relativistic quantum mechanics', would be to generalize the Schrödinger equation using the relativistic dispersion relation. There is a wavefunction giving the probability at time t to find particle 1 at x_1, particle 2 at x_2, and so on.

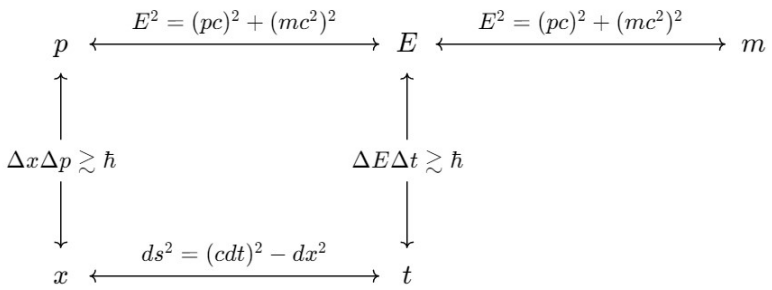
Fact 2: Relativistic quantum mechanics is inconsistent.
Let us return to our particle in a very small box, where Δp is ultrarelativistic, say many times the mass of our particle. Here we use a principle from SR, namely the dispersion relation. This tells us that ΔE

is also very large. Now, with high probability the energy of the system is much larger than the rest mass of the particle. In fact, there is now enough energy to spontaneously create more particles, so that the particle number is also probabilistic. This can also be seen using the energy-time uncertainty relation: if one stares at a patch of vacuum for a very short time, the notion of how many particles there are is an ill-defined notion. This is impossible to explain in relativistic quantum mechanics where particle number is constant, or maybe at best you might be able to extend the theory so there is some upper bound. We need something more, and this observation begins the high road to quantum field theory.

- Note: in QFT, the first mechanism is explained by the ability to rip particles straight out of the vacuum if the local energy is high enough. The energy-time problem is explained by virtual particles, which only 'exist' for short periods of time, popping in and out of existence. Far be it from static and boring, in QFT the vacuum is a swirling, boiling sea with profoundly rich dynamics.

1.2 Units

Dimensionful Quantities in QFT
In all of physics, there are 4 basic dimensionful units from which any dimensionful unit can be derived. These measure time $[T]$, length $[L]$, mass $[M]$, and temperature $[T]$. In this course we ignore temperature. In Newtonian mechanics, these are all unrelated. Let's see what happens in QFT. For convenience, let's also add p ($[MLT^{-1}]$) and E ($[ML^2T^{-2}]$).



We know that QFT = QM + SR. Each of the theories on the right is characterized by a fundamental dimensionful quantity. In special relativity, we have $c \sim 3 \cdot 10^8 m\, s^{-1}$. It determines the speed of causality relating the structure of space to the structure of time, and determines the dispersion relation relating momentum to mass-energy. In quantum mechanics, $\hbar \sim 1.1 \cdot 10^{-34} kg\, m^2\, s^{-1}$ determines the intensity of quantum probability fluctuations, demarcating the scale at which the sum over all histories in the path integral can be approximated by the principle of least action. From the perspective of the uncertainty relation, it relates position to momentum, or energy to time.

It figures then, that QFT should include both. All these quantities are now related. For instance, generically when we increase the energy E, we increase the momentum p of our particles, and have access to higher mass, heavier particles. By quantum mechanics, this means that characteristic times and distances involved in high energy processes are very short. For this reason, the field surrounding QFT is often called high energy physics.

Natural Units
There's a trick we can use to get these results for free just by introducing some notation. This is the idea of 'natural units', by which we write $\hbar = c = 1$. It's common when starting out to look at this and get confused. How can we set a dimensionful quantity equal to 1? This is not what we're doing, rather it is a shorthand which hides the dimension. What we're really doing is setting two types of measurements equal to one another: time and length. Measuring distance in time units is something all of us are familiar

with–for instance, 10 minutes as the crow flies, or distances between stars in light-years. Here we are doing nothing more than that; all we need is a reference velocity, which special relativity naturally provides (for reasons stated above). By the same argument, using the reference 'action' quantity and the uncertainty relation, we see that we can measure momentum as inverse length, or energy as inverse time.

We started with 3 independent dimensionful quantities, and related 2 pairs. Consequentially, in QFT we can express any quantity in terms of length (e.g. fm $= 10^{-15}m$), or as more often done, a mass scale (e.g. GeV). Working through units one can check that with $c = \hbar = 1$,

$$[M] = [E] = [P] = [L]^{-1} = [T]^{-1} \tag{1}$$

That is, momenta and accessible masses scale with energy, while distance or time scale inversely with energy. This is our result from before, which natural units and dimensional analysis have given us for free.

Note 1: For instance, we can talk about the length scale associated with a particle of mass m just by taking its inverse (in natural units). This is called the Compton wavelength, $\lambda_c = 1/m$. Physically, it's the wavelength of a photon with frequency equal to the rest energy of said particle.

Note 2: Restoring units. Given a generic measurement in terms of some energy scale, how do we return to normal units? This is done by multiplying our answer by some factors of c and $\hbar$ to give the desired dimensions. For instance, say we have a volume $V = 10^{-3}\text{GeV}^{-3} = 10^{-3}\hbar^m c^n \text{GeV}^{-3}$. Since volume has dimensions $[L]^3$, a straightforwards calculation gives $m = n = 3$. Using $1\text{GeV} \sim 10^{-10}J$ and substituting SI values of c and $\hbar$, we get a volume of 10^{-6}fm^3.

2 The Lorentz Group

In this section we review special relativity, at the core of which is the study of Lorentz transformations, which form the Lorentz group. You should think of a group as a bag of verbs, a set whose elements act on a system, such as translations or rotations. For instance, 'turn 30 degrees', or 'move 2 units to the right.' Each element must have an inverse, and you can compose elements by doing one after another.

The Lorentz group can be intimidating, even though as we'll see there's nothing to be intimidated about. To drive this point home we'll start by discussing rotations, which you should all be experts in. Lorentz transformations are just spicy rotations, and all said spice is contained in a single minus sign. If you ever get confused about Lorentz transformations, ask the same question about rotations.

2.1 Rotations

SO(2): Let's start with the simplest case. Consider the space of proper (orientation preserving) rotational symmetries in 2 dimensions. A point in the plane transforms as:

$$\begin{aligned} x &\to x\cos\theta + y\sin\theta \\ y &\to -x\sin\theta + y\cos\theta \end{aligned}$$

I can package this as a column or a row vector:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x & y \end{pmatrix} \to \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Index notation: we can write the rotation as a matrix acting on column vectors as $x^i \to R^i{}_j x^j$, and row vectors as $x_i \to x_j (R^T)^j{}_i$.

- Note 1: Einstein summation convention. If an index is repeated, it is assumed to be summed over. This is called a contraction.
- Note 2: Upper and lower indices. Column (contravariant) vectors always carry upper indices, while row (covariant) vectors always carry lower indices. Contractions are only permitted between an upper index and a lower index, reflecting matrix multiplication.
  This is a bookkeeping device that doesn't matter for rotations, and many authors are flippant about it. However, it is reflective of a deep mathematical fact. In general, matrices can act on vectors as a row or a column, and these actions are different. For example, this is why quarks are different from antiquarks. But when the matrix is real, as for rotations, these actions are the same.

We can check explicitly that in 2D, $R^T R = \mathbb{1}$, or in index notation $(R^T)^i{}_j R^j{}_k = \delta^i{}_k$.

SO(N): More generally, rotations act linearly on vectors such that distances between vectors are preserved. In Euclidean space, the distance between x^i and y^i is given by $\delta_{ij}x^i y^j$. Let us see what this means:

$$x_i y^i = \delta_{ij}x^i y^j \to \delta_{kl}R^k{}_i R^l{}_j x^i y^j = (R^T)_{il}R^l{}_j x^i y^j = x_i (R^T)^i{}_l R^l{}_j y^j$$

Since this holds for all x and y, this means that we have $N \times N$ matrices satisfying $R^T R = \mathbb{1}$, which is an equivalent way of characterizing rotations.

The metric: this is an inner product on a space, and provides a notion of distance. It is given by a symmetric matrix g_{ij}, with inner product defined using matrix multiplication, $\langle \mathbf{x}|\mathbf{y}\rangle = h_{ij}x^i y^j$. In Euclidean space, $h_{ij} = \delta_{ij}$. A rotation is a transformation preserving the metric: from the above we have

$$(R^T)_i{}^k \delta_{kl} R^l{}_j = \delta_{ij} \qquad \text{or} \qquad R^T \mathbb{1} R = \mathbb{1}$$

Discrete transformations: another linear transformation that preserves distance is the flip $(x^1, \ldots, x^n) \to (-x^1, x^2, \ldots, x^n)$. It is not orientation preserving, and one cannot reach it by performing strictly orientation preserving transformations. Adding the flip to our existing transformations gives O(N) instead of SO(N). O(2) has 2 connected components, and the component connected to the identity (trivial rotation) is SO(n).

Now, we could now more generally ask what a rotation really is, or study in greater detail the algebraic properties of the group of rotations. For us, this is the wrong question to ask. Rather, we ask the question of what we can rotate, i.e. on what kind of objects rotations can (linearly) act. Such a action is called a representation of SO(N). This may seem like a less fundamental question, but in physics any group that one encounters always acts on a system. A group is a bag of verbs. By introducing some unrelated math jargon, this approach can be summarized by the following maxim,

Groups, like men, will be judged by their actions, not their words.
This is why we should study their representations, not their presentations.

1. Scalars: objects that are invariant under a rotation. For instance, the number 3, the mass of a particle, or the action. They form a 0-dimensional space on which rotations act trivially.
2. Vectors: objects that transform as $v^i \to R^i{}_j v^j$, or $v_i \to v_j (R^T)^j{}_i$. These objects have 1 index, and examples include a position x^i, a derivative ∂_i, or a 3-momentum p^i of a particle.
3. Tensors: multi-index objects, where each index transforms as a row or column vector. For instance,

$$T^{i_1,\ldots,i_m}{}_{j_1,\ldots,j_n} \to R^{i_1}{}_{k_1} \ldots R^{i_m}{}_{k_m} R_{j_1}{}^{l_1} \ldots R_{j_m}{}^{l_n} T^{k_1,\cdots,k_k}{}_{l_1,\cdots\ldots,l_n}$$

4. Pauli spinors: these are more exotic objects for which we can define rotations. For SO(3), the rotation matrices are 2×2 acting on a 2D complex space, and look like $R_{ab} = e^{i\theta_i \sigma^i{}_{ab}}$. Here the θ's are constants, and σ^i's are the Pauli matrices.

This is it. A powerful result from representation theory is that any (finite dimensional) mathematical object that is 'rotatable', can be decompose into subobjects that behave as scalars, vectors, tensors, and spinors under rotations.
Note: we call objects that don't change under scalars 'invariant', while objects that transform as representations of SO(3) 'covariant'. We can contract indices of covariant objects (vectors, tensors, etc.) with each other to get something that is invariant, for instance $x_i p^i$ or $x^i \partial_i$.

Finally, we bring up fields. A field is a function that associates an object to each point in space. These too can be rotated, and we can mainly quote our previous results. The point of caution here is that there are two things that can be rotated–the field itself as a function over space, and the point in space at which it is evaluated. We want to see how the field behaves under the former.

1. Scalar fields: under a rotation, $\phi \to \phi'$. Scalar means that if we evaluate the transformed field at the transformed point $x' = Rx$, the field remains the same, $\phi'(x') = \phi(x)$. We can thus isolate the transformation property of the field:
$$\phi(x) \to \phi'(x) = \phi(R^{-1}x)$$
2. Vector fields: $V^\mu \to V'^\mu$. The transformation law is
$$V^i(x) \to V'^i(x) = R^i{}_j V^j(R^{-1}x)$$
The key feature is that it picks up a rotation matrix out front. Again, we need to undo the transformation of the point in space where the field is evaluated.
3. Tensor fields, spinor fields are obtained similarly.

2.2 Lorentz Transformations

We just spent an large amount of time discussing rotations. But this was not in vain; everything we saw can be extended with barely any effort. Lorentz transformations are just rotations, with some extra minus factors we need to keep track of. We denote the (connected part of the) Lorentz group in d spacetime dimensions as $\mathrm{SO}(1, d-1)$, with coordinates $x^\mu = (t, \mathbf{x}) = (t, x^i)$.

The metric: we know from special relativity that the invariant distance between 2 points in spacetime is given by $s^2 = -t^2 + \mathbf{x}^2$. s is the norm of x^μ with respect to the Minkowski metric,
$$g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$
The -1 contains all the difference between ordinary rotations and the Lorentz group. The inner product between vectors is $x \cdot y = g_{\mu\nu} x^\mu y^\nu$, and the norm of a vector is $x \cdot x = g_{\mu\nu} x^\mu x^\nu$.

Lorentz transformations: these are the set of symmetries of spacetime preserving the metric, or invariant distance. They act on spacetime vectors as matrices Λ^μ_ν, which as with ordinary rotations, must satisfy
$$\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \tag{2}$$
As with the group of ordinary rotations SO(4) in 4 dimensions, SO(1, 3) has 6 independent components, where we can 'rotate' any axis into any different axis: (t, x), (t, y), (t, z), (x, y), (x, z), (y, z). The latter 3 generate the rotations of 3D space. However, because of the -1 in the tt-component of the metric, transformations mixing a time and space axis are phenomenologically different. We call these boosts.

- Example: consider a boost between only x and t. These are transformations of x and t keeping the distance $t^2 - x^2 = \mathrm{cst}$. This generates motion along hyperbolas (as opposed to circles for rotations). Just like 2D rotations, we may write the most general such transformation explicitly as
$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \cosh\beta & \sinh\beta \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{3}$$

Here v is just a parameter. We recognize $\frac{1}{\sqrt{1-v^2}} = \gamma = \cosh\beta$, where β is the rapidity.

Aside: parameterizing the Lorentz group. Lie groups, Lie algebras, and the exponential map.

- It is not difficult to write rotations or boosts between any 2 axes. When you compose them, however, it is much easier to use a matrix exponential to write a generic Λ. I could have also done this for ordinary rotations–this provides a profound mathematical alternative to the mess that is the Euler angles. For the Lorentz group one has:

$$\Lambda = \exp(i\beta_i K_i + i\theta_i J_i) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = \lim_{n\to\infty}\left(\mathbb{1} + \frac{1}{n}\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)^n \tag{4}$$

  Here K_i, J_i are antisymmetric 4×4 matrices generating boosts and rotations respectively, with non-vanishing components:

$$(J_i)_{jk} = -i\epsilon_{ijk} \qquad (K_i)_{0j} = \delta_{ij} = -(K_i)_{j0}$$

  They can be packaged nicely as a tensor of 4×4 matrices $\mathcal{J}^{\mu\nu}$. Furthermore, $\omega_{\mu\nu}$ can be related to the rapidities and rotation angles from before:

$$\mathcal{J}^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} \qquad \omega_{\mu\nu} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

- In the last equality of (4), we have used the exponential map to write a general Lorentz transformation as the composition of infinitesimal Lorentz transformations, meaning they are infinitesimally close to the identity. Intuitively this makes sense, I can add infinitesimal transformations to get something finite. The consequences of this are hefty–in particular, I can know almost everything about the Lorentz group (as in, aside from some topological data) just by studying the group in an infinitesimal neighborhood of the identity.
  A more familiar example might be the space of 2D rotations SO(2), which is just a circle parameterized by the rotation angle. Very close to the identity ($\theta = 0$), this just looks like a line.
- More generally, the rotation and Lorentz groups are special cases of Lie groups, which are groups with continuous parameters. The Lie algebra is the tangent space of the Lie group at the identity, and you can always define an exponential which maps the Lie algebra to the Lie group. These form a core area of study in high energy theory.

Discrete transformations: so far, we have only looked at $\mathrm{SO}(1,3)$ which is the part of the Lorentz group connected to the identity. This called the proper orthochronous subgroup. Just like with the rotation group, the Lorentz group is disconnected, but it has 4 disconnected components instead of 2. From $\mathrm{SO}(1,3)$ one can access the other 3 by acting with parity $P : (t,x,y,z) \to (t,-x,-y,-z)$ or time-reversal $T : (t,x,y,z) \to (-t,x,y,z)$. Together, P and T generate the set $\{1, P, T, PT\}$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

As before, we can consider objects which can be Lorentz transformed–that is, both rotated and boosted. For scalars, vectors, and tensors, the analysis proceeds identically, just replacing Roman indices with Greek ones, and $R^i{}_j$ with $\Lambda^\mu{}_\nu$. Examples of scalars include a particle's mass or the action. Vectors include $x^\mu = (t, x^i)$, $p^\mu = (E, p^i)$, and $\partial^\mu = (\partial^t, \partial^i)$. The discussion for scalar, vector, and tensor fields also proceeds identically. If you're not familiar with it, it's a good exercise to go over everything we did for rotations, and write it for Lorentz transformations. As a final note, the spinors and spinor fields we get are more tricky; the resulting objects that can be rotated and boosted are called Weyl spinors, which differ a bit from Pauli spinors. You'll see much more on spinors in the weeks to come.

3 Why Fields?

I'd like to end off today by answering one of the big questions which is often obscured. This is to address the F in 'QFT'. Where do the fields come from? They are not found in the axioms of either QM or SR. The claim is the following: any theory combining both QM and SR describing even just 1 particle necessitates a description using fields. Here we outline the argument.

1. Special Relativity and Symmetries
   From SR we have the symmetries of our theory: rotations, boosts, and translations. Physics should be the same regardless of whether I do it here or turn 30 degrees walk 2 meters, and jump aboard a train going half the speed of light. The group that combines all of these is called the Poincaré group, which I get by adding translations to the Lorentz group.

2. Describing Particles
   We know that our universe has particles, with momentum, mass, spin, and all sorts of quantum numbers. If we rotate or boost to change frame, only the momenta and spin component (say, S_z) change, and both change in a Poincaré-invariant way. The other quantum numbers are invariant. This motivates the definition of a particle a set of states that mix only among themselves under Poincare transformations.

3. Quantum Mechanics and Representations
   In QM, the states representing a particle form a vector space, and we have argued that the Poincaré group must act naturally on it, $|\psi\rangle \to \mathcal{P}|\psi\rangle$. A set of objects $|\psi\rangle$ which mix under the action of a group is a representation. We saw some representations of this group earlier: scalars, vectors, tensors, and spinors. These are finite-dimensional representations, with dimensions 1, 4, 16, etc. We saw how they transform under rotations and boosts, and they are invariant under rotations. So far this is fine. We can describe particles without fields. The moral here is that 'particles transform under irreducible representations of the Poincare group.' Weinberg uses this as the definition of what a particle is.

4. Unitarity
   But when we add one more condition required from quantum mechanics, we will see that we need fields. This is the notion of unitarity. In particular, we need matrix elements to be invariant under Poincaré transformations:

   $$\mathcal{M} = \langle\psi_1|\psi_2\rangle = \langle\mathcal{P}\psi_1|\mathcal{P}\psi_2\rangle = \langle\psi_1|\mathcal{P}^\dagger\mathcal{P}|\psi_2\rangle$$

   This requires $\mathcal{P}^\dagger\mathcal{P} = \mathbb{1}$, i.e. the matrices which encode the action of Poincaré transformations must be unitary. The problem lies in that there exist no finite-dimensional unitary representations of the Poincaré group. Therefore, if we assume unitarity, particles cannot be described the same way as traditional quantum mechanics.

5. Wigner's classification
   Eugene Wigner classified all the unitary Poincaré representations all the way back in 1939. All of these are infinite dimensional, and have natural descriptions in terms of fields. Neither I nor Hong will go through this proof, you can find it in Weinberg Chapter 2 if you're interested. The key lies in that the group acts on the field's spacetime dependence in a way which restores unitarity. But we have reached the moral I'd like all of you to take away, which is that any theory combining both QM and SR describing particles necessitates a description using fields.

Recitation 2

Wentao Cui

February 17, 2023

1 Quantization in Quantum Mechanics

1.1 First, there was Hamilton

Let us review classical mechanics, in the Hamiltonian formalism:

- States: given by a point in phase space $M = \mathbb{R}^{2d}$ with coordinates (q_j, p_j) for $1 \leq j \leq d$
- Observables: functions on phase space, $F(M)$
- Dynamics: states evolve according to Hamilton's equations, $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

We can capture the dynamics by defining the Poisson bracket

$$\{\cdot,\cdot\} : F(M) \times F(M) \to F(M), \qquad (f_1, f_2) \mapsto \{f_1, f_2\} = \frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q_i}$$

- Mathematically, it gives $F(M)$ a group structure (more precisely, a Lie algebra structure, due to antisymmetry + Jacobi identity)
- The Poisson bracket generates time translations: an observable evolves in time

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t} = \{A, H\} + \frac{\partial A}{\partial t}$$

  where the second term vanishes if A does not explicitly depend on time
- Observe: this looks a lot like the equations of motion for operators in the Heisenberg picture

$$\frac{d\mathcal{O}}{dt} = -i[\mathcal{O}, H] + \frac{\partial \mathcal{O}}{\partial t}$$

1.2 Quantization

Quantization: the procedure of turning a classical theory into a valid quantum theory

- Many quantization schemes in QM: Dirac quantization, path-integral quantization, deformation quantization, geometric quantization, etc. Can generalize each of these to QFT.

Axiom: whenever a group G acts on a classical system, in the corresponding quantum theory, the Hilbert space $\mathcal{H}$ carries a unitary representation of G (have 'operators')

Dirac quantization: promote each classical observable f to an self-adjoint operator $\hat{O}_f$ acting on a Hilbert space $\mathcal{H}$ such that $\hat{O}_{\{f,g\}} = -\frac{i}{\hbar}[\hat{O}_f, \hat{O}_g]$ (replace P.B. with commutators)

- If we view $F(M)$ as a group with operation given by the Poisson bracket, quantization gives a unitary representation on $\mathcal{H}$
- For positions and momenta, we have:

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar$$

We can try continue this prescription to more complicated functions of q and p, but this breaks down for polynomials of degree ≥ 3 (Groenewold's theorem). In particular, operator ordering ambiguities generate corrections of higher-order in h. To second order in q and p said ordering ambiguities are not fatal, making quantities like the Hamiltonian well-defined.

Usually, we only care about restricting quantization to linear (sometimes quadratic) polynomials in x, p

- In classical mechanics, linear functions in q and p are closed under taking Poisson brackets, generating the Heisenberg group
- In quantum mechanics, we want operators satisfying (taking $\hbar = 1$)

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i$$

Existence and uniqueness: given by the Stone-von Neumann theorem

The Stone-von Neumann theorem:

1. There are no finite dimensional representations of the Heisenberg group
   I.e. $\hat{x}$ and $\hat{p}$ cannot be finite dimensional matrices (proof: take trace)
2. All (irreducible) representations of $\hat{x}$ and $\hat{p}$ are unitarily equivalent, and act on the Hilbert space of square-integrable functions $L^2(\mathbb{R})$

This is the most important theorem in quantum mechanics. In particular (2) proves that the following are all equivalent.

- Position-space representation: $\hat{x} \to x, \quad \hat{p} = -i\frac{\partial}{\partial x}$
- Momentum-space representation: $\hat{x} \to i\frac{\partial}{\partial x}, \quad \hat{p} = p$ related by Fourier (unitary!)
- We have a representation at every t. Time evolution, then, relates representations at different times, $(\hat{x}(t), \hat{p}(t), \mathcal{H}(t))$. By Stone-von Neumann all representations of $\hat{x}$ and $\hat{p}$ are unitarily equivalent, so time-evolution must be unitary. There is an inherent ambiguity: we can either evolve the operators (active), the states (passive), or something in-between, but all these are fundamentally the same. This demonstrates the equivalence of the Heisenberg and Schrödinger pictures of QM.

In practice, to quantize a Lagrangian system we do the following.

1. Identify the generalized coordinates (degrees of freedom) q_i of a classical system
2. Derive the conjugate generalized momenta p_i from the Lagrangian.
3. Promote (q_i, p_i) into operators $(\hat{q}_i, \hat{p}_i)$, which satisfy the canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\{q_i, p_j\} = i\hbar\delta_{ij}$$

Example: the classical harmonic oscillator, $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$ (e.o.m. $\ddot{x} - \omega^2 x = 0$)

- Conjugate momentum: $p = \dot{x}$, Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$
- We can introduce 'normal coordinates' $a = \frac{1}{\sqrt{2\omega}}(\omega x + ip), \quad a^* = \frac{1}{\sqrt{2\omega}}(\omega x - ip), \quad H = \omega a a^*$
- Quantization: can promote (x, p) to operators, or (a, a^*) to operators. Suppose we do the latter. the commutator is given by the Poisson bracket

$$[\hat{a}, \hat{a}^*] = i\hbar\{a, a^*\} = i\hbar\left(\frac{\partial a}{\partial x}\frac{\partial a^*}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial a^*}{\partial x}\right) = i\hbar(-i) = 1$$

- The operators $\hat{a}$, $\hat{a}^\dagger$ naturally act on the Bargmann-Fock space of square-integrable holomorphic (complex differentiable) functions. By Stone-von Neumann, there must be a unitary map between this and $L^2(\mathbb{R})$ in the position representation. Indeed there is, you should think of this map as just a change of basis. This map identifies Bargmann-Fock as the space of coherent states. Mathematically, it's nice to work in the basis of coherent states, because unlike with delta-functions and plane waves, I never encounter any problems with infinities or normalizability.

2 Quantization in QFT

2.1 Classical Field Theory

We review some basic field theory. We work in Minkowski space, with a scalar field ϕ. Everything in this discussion can be extended to vector, tensor, and spinor fields with some effort (sometimes, with a lot of effort), which will be the subject of much of this course.

The starting point is the action. This is the time-integral of the Lagrangian L, or the spacetime-integral of the Lagrangian density $\mathcal{L}$.

$$S[\phi] = \int dt L[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

What it means for ϕ to be a Lorentz scalar is that under a Lorentz transformation Λ, the field transforms as $\phi \to \phi'$, such that $\phi'(\Lambda x) = \phi(x)$. This is a confusing point, so I'll elaborate. [draw picture]

The principle of least action leads to the Euler-Lagrange equations of motion, which governs the time-evolution of ϕ.

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

Symmetries and Noether's theorem. Ask class if they want to go over an example. Use 2-component scalar field ϕ_a with SO(2) symmetry. Angular momentum.

To prepare for quantization, let us pick a specific model. Namely, we work with a free (zero potential) real scalar field of mass m, with Lagrangian and equation of motion given by

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - m^2\phi^2, \qquad (\partial^2 - m^2)\phi(x) = 0$$

The conjugate momentum to ϕ is $\pi = \partial_t \phi = \dot{\phi}$. The Hamiltonian density is thus

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

In order to determine the degrees of freedom of our system, let us examine the equations of motion, which we write in Fourier (momentum) space $\tilde{\phi}(t, \mathbf{k})$:

$$(-\partial_t^2 + \omega_{\mathbf{k}}^2)\tilde{\phi}(t, \mathbf{k}) = 0$$

Here $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$, and $\mathbf{k}$ a free parameter. The equation of motion for our field has now been decoupled into its momentum components. For each $\mathbf{k} \in \mathbb{R}^3$ we have such an equation, which is precisely the equation of motion for a harmonic oscillator of frequency $\omega_{\mathbf{k}}$. The moral is that now we can immediately read off our system's degrees of freedom: for each (continuous) momentum value we get a harmonic oscillator. This is a generic feature of free QFTs.

2.2 Canonical Quantization

We are ready to quantize our theory, and in fact we can immediately quote our results from earlier for a single harmonic oscillator. The mode $\tilde{\phi}(t, \mathbf{k})$ becomes an operator $\hat{\tilde{\phi}}(t, \mathbf{k})$ given by

$$\hat{\tilde{\phi}}(t, \mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}}(t) + \hat{a}_{\mathbf{k}}^\dagger(t)) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \hat{a}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} t})$$

To get back to position space we take a Fourier transform to yield a mode expansion:

$$\hat{\phi}(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{k}}}(\hat{a}_\mathbf{k}e^{-i\omega_\mathbf{k}t+i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{k}e^{i\omega_\mathbf{k}t-i\mathbf{k}\cdot\mathbf{x}})$$

This is an important equation in QFT. We can also compute the canonical momentum:

$$\hat{\pi}(t,\mathbf{x}) = \partial_t\hat{\phi}(t,\mathbf{x}) = -i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{\omega_\mathbf{k}}{2}}(\hat{a}_\mathbf{k}e^{-i\omega_\mathbf{k}t+i\mathbf{k}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{k}e^{i\omega_\mathbf{k}t-i\mathbf{k}\cdot\mathbf{x}})$$

Finally, quantization. There are two ways we can do this.

1. Demanding the canonical commutation relations for a generalized coordinate and conjugate momenta, in this case the pair ϕ and π:

$$[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{x}')] = [\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = 0, \qquad [\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

2. Using the above insight where we have a continuum of harmonic oscillators we want to quantize:

$$[\hat{a}_\mathbf{k},\hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_\mathbf{k},\hat{a}^\dagger_{\mathbf{k}'}] = 0, \qquad [\hat{a}_\mathbf{k},\hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

   The prefactor $(2\pi)^3$ is not important, since it can always be removed by rescaling operators. However, it is necessary so that (1) and (2) are consistent, as one can check using the mode expansions for $\hat{\phi}(x^\mu)$ and $\hat{\pi}(x^\mu)$. In fact, (1) and (2) are equivalent in that imposing each commutator implies the other, serving as a good consistency check of our theory.

By imposing the necessary canonical commutation relations, we have thus quantized a classical field theory, paving the way for quantum field theory.

It remains to discuss the Hilbert space of our theory, and again we draw from our knowledge of the quantum harmonic oscillator. $\mathcal{H}$ is generated by acting creation operators $\hat{a}_\mathbf{k}$ on some vacuum state $|0\rangle$. There is an important difference: in QM creation operators create one unit of energy quanta. In QFT, a creation operator $\hat{a}^\dagger_\mathbf{k}$ creates a bona fide particle of momentum $\mathbf{k}$. The total Hilbert space is thus

$$\mathcal{H} = \bigoplus_{n\geq 0}\mathcal{H}_n, \qquad \mathcal{H}_n = \text{span}\{\hat{a}^\dagger_{\mathbf{k}_1}\cdots\hat{a}^\dagger_{\mathbf{k}_n}|0\rangle\}$$

where $\mathcal{H}_n$ is the n-particle sector, and $|0\rangle$ is the vacuum state (Note that $\mathcal{H}_0 = \{|0\rangle\}$). When we do quantum mechanics without QFT, we are locked into working in a particular sector $\mathcal{H}_n$. On the other hand, QFT provides a natural way to move between sectors: particle creation and annihilation are intrinsic to the theory.

A last note. In quantum mechanics, we have position operators $\hat{x}$, but no time operator t. In quantum mechanics time serves a privileged role as a parameter. Special relativity tells us that space and time are intimately related, and so in QFT we demote position to the same status as time, from an operator to just a parameter. Indeed, fields take values in both position and time. There is another way to approach particle physics. Namely, we could have also promoted time to an operator and retain position operators so they are again on equal footing. Doing so, one encounters several problems. For instance, owing to the -1 in the Minkowski metric, the Hilbert space no longer has a positive definite inner product. This is very bad, but not fatal–the theory can be remedied by restricting to a 'physical' Hilbert space. One can proceed to get a consistent quantum theory, but this is hard, less intuitive, and not particularly insightful for us. However, the approach of promoting time to an operator is how string theory is formulated, at least perturbatively.

2.3 The Role of Time in Quantization

I just told you that time and space should be treated equally, since they are part of the same entity we call spacetime. I will proceed to tell you why the opposite is true: even in QFT, time has a very privileged role to play.
Let us return to our scalar equation of motion,

$$(-\partial_t^2 + \omega_{\mathbf{k}}^2)\tilde{\phi}(t, \mathbf{k}) = 0$$

This is a second order ODE, so for a given $\omega_{\mathbf{k}}$ this has 2 solutions: $\tilde{\phi}(t, \mathbf{k}) \propto e^{\pm i\omega_{\mathbf{k}} t}$. One has positive frequency, and the other negative frequencies: it is this which demarcates particles from antiparticles. To be more precise, it is this ∂_t operator whose eigenvalues separates modes into positive and negative frequency. In the language of general relativity, this separation depends on the existence of a global timelike Killing vector, which is a coordinate transformation which preserves the metric. In special relativity we work in Minkowski space, where ∂_t fulfills this role.

More generally, we can quantize a field theory in curved spacetimes that lack a timelike Killing vector. There is thus no natural decomposition of ϕ into positive and negative frequencies, into particles and antiparticles, into creation and annihilation operators. In fact, the notion of a particle in curved space does not have universal significance, and is observer-dependent. A state where one observer observes no particles, another observer might see them. Because the creation and annihilation operators are crucial to defining the vacuum of the theory, different observers will have different vacua. Generically, such vacua are no longer devoid of particles, even if an observer is free-falling. In Minkowski space we don't have this problem because the vacuum state we defined earlier today is the agreed vacuum for all inertial observers, as both are invariant under the Poincaré group.

Let me end off today with 2 experimental consequences of formulating QFT in curved space.

1. Consider 2 observers in Minkowski space, with a scalar field. The first is inertial, and the second is uniformly accelerating at a. Suppose the field is in the vacuum state for the inertial observer, which is just the $|0\rangle$ state we wrote above. Then, the accelerating observer will see a thermal bath of particles, at temperature $T \propto a$. This phenomenon is called Unruh radiation, with T the Unruh temperature.

2. Consider a scalar field in a universe that is asymptotically Minkowski at early and late times, but undergoes a period of inflation from t_1 to t_2. Then, even if the field ϕ is prepared in the vacuum state at time $t = -\infty$, an inertial observer at late times will see particles: this is the phenomenon of particle creation is cosmological spacetimes.

Recitation 3

Wentao Cui

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1 Contour Integration

1.1 The Residue Theorem

It's commonly said in math that the shortest path between 2 points in the real domain passes through the complex plane. One such path is contour integration, a powerful technique using the tools of complex analysis that can be applied to evaluate difficult real integrals. It has incredible importance in QFT for reasons I'll discuss, so I'd like to start by reviewing it.

Simple Poles

First, consider a function f, holomorphic in some region $D \subset C$ with $C = \partial D$. **Draw figure.** Then Cauchy's integral theorem tells us that

$$\oint_C dz f(z) = 0$$

This is a corollary of Green's theorem in the plane, combined with the Cauchy-Riemann equations.

We take something more interesting, take $g(z) = \dfrac{f(z)}{z - z_0}$, where $f(z)$ is holomorphic. There are 2 cases.

1. $z_0 \notin D$. Then g is holomorphic on D, so by Cauchy's theorem above the integral vanishes
2. $z_0 \in D$. **Draw figure.** By Cauchy the integral only picks up a contribution at the pole. Here the numerator is regular, and we can compute the integral of the denominator using the curve $z(t) = z_0 + \epsilon e^{it}$. We then have

$$\oint_C \frac{f(z)}{z - z_0} = 2\pi i f(z_0) =: 2\pi i \text{Res}_g(z_0)$$

The story is very similar if there are many poles in D: we can split up the domain of integration and evaluate each pole using the technique above. **Draw figure.** For instance,

$$h(z) := \frac{f(z)}{(z - z_0)\cdots(z - z_n)} = \frac{1}{z - z_0}\frac{f(z)}{(z - z_1)\cdots(z - z_n)} = \frac{1}{z - z_0}\text{Res}_h(z_0) = \frac{1}{z - z_k}\text{Res}_h(z_k)$$

$$\oint_C dz h(z) = 2\pi i \sum_k \text{Res}_h(z_k)$$

The way to intuitively understand the residue of a function at some point z_i is what you get when you block out the pole at z_i, and evaluate the rest of the function there.

Higher Order Poles

Technically this is only true for simple poles; for higher order poles (e.g. $1/(z-z_0)^2$), the story is a bit more complicated. For an nth-order pole, one has the formula

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}((z-z_0)^n f(z))$$

For simple poles $n = 1$ this gives what we expect,

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

What does this mean? One can expand a function f in terms of a Laurent series about a pole of order n. This is just a Taylor series with negative exponents, to account for the singular structure.

$$f(z) = \sum_{k \geq -n} \frac{a_k}{(z-z_0)^n}$$

Then, the residue of f is the coefficient of the $(z-z_0)^{-1}$ term–again, it's what happens when you block out the pole at z_i. One can now make sense of this formula. Say f has a pole of order n–how do we isolate the -1 term of the Laurent series when there are terms more singular? There is a trick: multiply the series by $(z-z_0)^n$, so we have no more singularities. Then, differentiate the regular series $n-1$ times so that terms corresponding to higher singular behavior vanish. The constant term is now just the coefficient of the -1 coefficient (with some combinatorial factor). Finally, one evaluates at $z = z_0$ to kill off every other regular term, isolating a_{-1}.
Usually you will only have to work with simple poles, but this formula is good to know.

Now, we can state the residue theorem in its full generality. Let $h(z)$ be some function, analytic except at a discrete collection of poles. Pick some closed curve C bounding a region D. One has

$$\oint_C dz h(z) = 2\pi i \sum_{z_j \in D} \text{Res}_h(z_j)$$

1.2 Applications

Example 1
Let's do an example. Consider the following real integral. The trick is to write this as a complex integral.

$$I = \int_{\mathbb{R}} \frac{e^{ix}}{x^2+1} = \int_{\mathbb{R}} dx \frac{e^{ix}}{(x+i)(x-i)}$$

This has poles at $x = \pm i$, with residues $\pm e^{\mp 1}/2i$.

We want to use the residue theorem. However, right now we can't because this requires a closed path of integration. There is a trick to get it of this form, known as closing the contour.

- We consider the semicircle of radius $R \gg 1$ in the upper half plane, going counterclockwise, $C = L_R \cup C_R$. **Draw figure.** By the residue theorem,

$$\oint_{L_R \cup C_R} dz \frac{e^{iz}}{z^2+1} = 2\pi i \text{Res}(i) = \frac{\pi}{e}$$

- We now take $R \to \infty$:

$$\lim_{R \to \infty} \oint_{L_R \cup C_R} dz \frac{e^{iz}}{z^2+1} = \oint_{\mathbb{R}} dz f(z) + \lim_{R \to \infty} \int_{C_R} dz f(z) = \frac{\pi}{e}$$

This is almost what we want, but the presence of this second term is annoying. Let's study it. for $f(z) = e^{ix}/(x^2+1)$, the integrand is exponentially suppressed along this contour, and vanishes:

$$\lim_{R\to\infty}\int_{C_R} dz f(z) \sim \lim_{R\to\infty}(\pi R)\frac{e^{-R}}{R^2} = 0$$

This is why we closed the contour upwards: if we closed in the other direction, the integrand would have been exponentially amplified, making our method useless.

We have recovered $I = \pi/e$.

We can summarize this 3-step procedure for integrals $\int_{\mathbb{R}}$. First, identify the poles and residues. Next, identify if closing upwards or downwards will give a vanishing contribution. Lastly, close in this direction and use the residue theorem.

Example 2
Now let's do an integral in QFT which is extremely important:

$$\Pi(x-y) = i\int đ^4p \frac{e^{ip\cdot(x-y)}}{p^2+m^2-i\epsilon}$$

Here $\epsilon > 0$ is a generic, arbitrarily small number. In particular, we'll see why this $i\epsilon$ is so important in QFT. Let's look for the poles of the integrand. We have

$$p^2+m^2-i\epsilon = -(p^0)^2+\mathbf{p}^2-i\epsilon = -(p^0)^2+\omega_{\mathbf{p}}^2-i\epsilon = (p^0+\omega_{\mathbf{p}}-i\epsilon)(-p^0+\omega_{\mathbf{p}}-i\epsilon) = 0$$

Where in the last step we don't care about quadratic terms, and renamed $2\omega_{\mathbf{p}}\,\epsilon \to \epsilon$. It is important when doing this that $\omega_{\mathbf{p}} > 0$: the positive sign of ϵ is sacred. Therefore, the poles are located at $p^0 = \pm\omega \mp i\epsilon$. **Draw figure.**

$$\Pi(x-y) = -i\int đ^3\mathbf{p}e^{i\mathbf{p}\cdot(\mathbf{x}\cdot\mathbf{y})}đp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0+\omega_{\mathbf{p}}-i\epsilon)(-p^0+\omega_{\mathbf{p}}-i\epsilon)}$$

The dp^0 integrand f has $\mathrm{Res}_f(-\omega_{\mathbf{p}}+i\epsilon) = \frac{1}{2\pi}\frac{1}{2\omega_{\mathbf{p}}}e^{i\omega_{\mathbf{p}}(x^0-y^0)}$, and $\mathrm{Res}_f(\omega_{\mathbf{p}}-i\epsilon) = \frac{1}{2\pi}\frac{1}{2\omega_{\mathbf{p}}}e^{-i\omega_{\mathbf{p}}(x^0-y^0)}$

Now we close the contour. We want exponential decay in $e^{-ip^0(x^0-y^0)}$, so the half-plane we pick actually depends on $\mathrm{sign}(x^0-y^0)$.

1. $x^0 > y^0$. We close down, picking up the pole at $\omega_{\mathbf{p}} - i\epsilon$:

$$I(x^0-y^0) = 2\pi i\frac{1}{2\pi}\frac{1}{2\omega_{\mathbf{p}}}e^{-i\omega_{\mathbf{p}}(x^0-y^0)} = \frac{i}{2\omega_{\mathbf{p}}}e^{-i\omega_{\mathbf{p}}(x^0-y^0)}$$

2. $x^0 < y^0$. We close up, picking up the pole at $-\omega_{\mathbf{p}} + i\epsilon$. A similar calculation shows

$$I(x^0-y^0) = \frac{i}{2\omega_{\mathbf{p}}}e^{-i\omega_{\mathbf{p}}(y^0-x^0)}$$

Finally, we can substitute this into $\Pi(x-y)$ to get our result:

$$\Pi(x-y) = \begin{cases} \int \dfrac{đ^3\mathbf{p}}{2\omega_{\mathbf{p}}}e^{ip\cdot(x-y)}|_{p^0=\omega_{\mathbf{p}}}, & x^0-y^0 > 0 \\ \int \dfrac{đ^3\mathbf{p}}{2\omega_{\mathbf{p}}}e^{ip\cdot(y-x)}|_{p^0=\omega_{\mathbf{p}}}, & x^0-y^0 < 0 \end{cases}$$

We will shortly see how this is equivalent to the time-ordered 2-point function, $\langle 0|T\phi(x)\phi(y)|0\rangle$.

2 Green's Functions

When learning QFT, it can be difficult to grasp the true value of Green's functions. I'd like to make this very clear in the following statement, which we'll spend some time trying to understand. Vacuum expectation values of products of free field operators can be written in terms of Green's functions of the wave equation (equation of motion). Today I'd like to give you some intuition playing around with these things, and relate them to the contour integrals we just talked about.

There are a lot of Green's-adjacent functions we can write down, and here are some of them. Note that these expressions depend on 4-vectors x and x', which can have different time components.

- Wightman functions $G^+(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle, \qquad G^-(x,x') = \langle 0|\phi(x')\phi(x)|0\rangle$
- Schwinger/Hadamard functions $iG(x,x') = \langle 0|[\phi(x),\phi(x')]|0\rangle, \qquad G^{(1)}(x,x') = \langle 0|\{\phi(x),\phi(x')\}|0\rangle$
- Feynman (T-ordered) function $iG_F(x,x') = \langle 0|T\phi(x)\phi(x')|0\rangle = \theta(t-t')G^+ + \theta(t'-t)G^-$
- Retarded/advanced function $G_R(x,x') = -\theta(t-t')G(x,x'), \qquad G_A(x,x') = \theta(t-t')G(x,x')$

The fact that there are so many of basically the same thing, and that all of them have the names of some of the most important figures in QFT, should tell you how important these are.

Using the Klein-Gordon equation $(\partial^2 + m^2)\phi(x) = 0$, we can see that $G^\pm$, and consequentially G, $G^{(1)}$ satisfy the homogenous equation

$$(\partial_x^2 + m^2)\mathscr{G}(x,x') = 0$$

We can also evaluate the Klein-Gordon operator on the Feynman Green's function:

$$(\partial_x^2 + m^2)G_F(x,x') = (\partial_x^2 + m^2)(\theta(t-t')G^+(x,x') + \theta(t'-t)G^-(x,x'))$$

It is very instructive to do this calculation. On each term, the derivative can act in 3 ways by the product rule. If both ∂_x act on G^+ or G^-, the term added to the m^2 term vanishes by the equation of motion. The other terms can be evaluated by $\partial_t\theta(t-t') = \delta(t-t')$ and the equal time commutation relations to find the Feynman Green's function solves the inhomogenous Klein-Gordon equation with δ-function source.

$$\begin{aligned}(\partial_x^2 + m^2)G_F(x,x') &= -\delta^{(4)}(x-x') \\ (\partial_x^2 + m^2)G_{R,A}(x,x') &= \delta^{(4)}(x-x') \qquad \text{(similar computation)}\end{aligned}$$

In this way the Feynman, retarded, and advanced functions are 'true' Green's functions, while the Wightman, Schwinder, and Hadamard functions are not, being solutions of the homogenous equation of motion. In the general way we can build general solutions to the inhomogenous equations of motion out of these Green's functions.

2.1 Boundary Conditions

This begs the question: what is the difference between all of these Green's functions? The answer lies in the boundary condition.

Observe that due to the presence of the $\theta(t-t')$, the retarded function $G_R(x,x') = -\theta(t-t')(G^+ - G^-)$ vanishes unless $t > t'$. Looking at the equation of motion, the Green's function measures the response to a δ-function which stimulates the field at $t = t'$. We could have instead chosen time to run backwards: the

Klein-Gordon equation is agnostic, since it is invariant under $t \to -t$. Doing so gives you the advanced Green's function: instead of everything travelling forwards in time, the field disturbances travel backwards.

The Feynman function is an average of the retarded and advanced solution. In QFT this is the right thing to do because the positive and negative frequency solutions to the equations of motion are on equal footing. That is, we have particles AND antiparticles, which can be mathematically treated as particles travelling backwards in time. Causality must be imposed on both. If I disturb my particle field at time t, the response at time t' will be zero until $t' > t$. For the antiparticle, the reverse is true.

To summarize: the Feynman, retarded, and advanced functions describe the propagation of field disturbances subject to certain boundary conditions. In QFT, causality, along with the existence of antiparticles, make the Feynman Green's function the right one to use.

2.2 Integral Representations

Integral representations for Green's functions can be obtained by substituting the mode decomposition of $\phi(x)$ and $\phi(x')$ into the definitions, and using properties of creation and annihilation operators. Let's do this for the Feynman Green's function, first for $t > t'$.

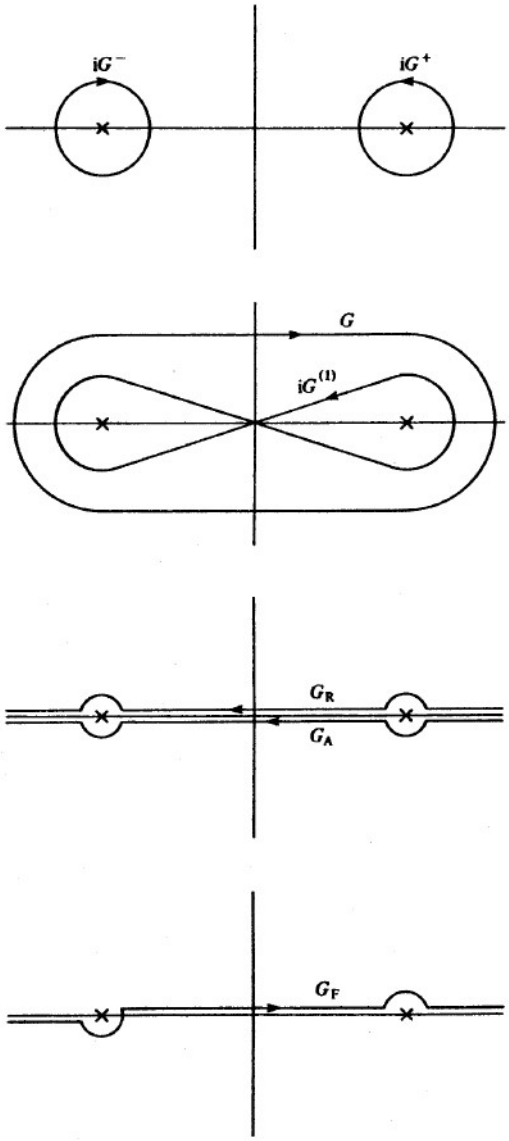
$$\begin{aligned} G^+(x,x')|_{t>t'} &= \langle 0|\phi(x)\phi(x')|0\rangle = \int đ^3\mathbf{k} đ^3\mathbf{k}' \frac{1}{(2\omega_\mathbf{k} 2\omega_{\mathbf{k}'})^{1/2}} e^{-i\omega_\mathbf{k} t + i\mathbf{k}\cdot\mathbf{x}} e^{i\omega_{\mathbf{k}'} t' i\mathbf{k}'\cdot\mathbf{x}'} \langle 0|a_\mathbf{k} a^\dagger_{\mathbf{k}'}|0\rangle \\ &= \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} e^{-ik\cdot(x-x')}, \qquad k^0 = \omega_\mathbf{k} \\ G^+(x,x')|_{t<t'} &= \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} e^{-ik\cdot(x'-x)}, \qquad k^0 = \omega_\mathbf{k} \end{aligned}$$

But this should be familiar! In fact, this was the answer of something we computed earlier! We thus have

$$G_F(x,x') = \Pi(x-y) = i\int đ^4p \frac{e^{ip\cdot(x-y)}}{p^2+m^2-i\epsilon}$$

To the Feynman Greens function we have associated a very specific integration contour, specified by the $i\epsilon$ prescription. Note that because $i\epsilon$ is so small, we can either deform the contour away from the real line and keep the poles real, or deform the poles away from the real line and integrate over $\mathbb{R}$.

We can do the exact same thing for the advanced and retarded Greens functions, and compare them to the value of $i\int đ^4p \frac{e^{ip\cdot(x-y)}}{p^2+m^2}$ with different choices of integration contour. To summarize: we can obtain each Greens function by performing some contour integral, with a specific choice of integration contour specifying the boundary condition of our theory. They are given below. **Draw figure.**



We may have done something quite peculiar to you today. We started with certain objects we like to study, these 2-point correlators, that are initially real-valued in position coordinates. But the choice of boundary conditions has imposed a specific complex (or pole) structure in our theory, and so we add complex analysis to our QFT toolbox. By considering analytic continuations, these objects have a very profound analytic structure universal to all QFTs. This is an incredibly deep rabbit hole that has birthed entire fields, none of which we'll have the time for.

I will mention one mindblowing (to me) application. We've discussed that in QFT, the Feynman Green's function is the right one to use. By the residue theorem, no integrals change if we deform our contour, so long as we don't cross any poles. We can therefore deform the contour from the real plane to the imaginary plane, and therefore compute integrals like the ones we've seen by integrating over purely imaginary time. If time is imaginary, then our metric becomes $(-i^2, +, +, +) = (+, +, +, +)$ which is now Euclidean. This is known as a Wick rotation. In this way, we can compute quantities in a Minkowski theory by converting to a theory in Euclidean space, which is essentially just a statistical field theory at zero-temperature. That last part requires more explanation, which I'll provide next week when you start talking about path-integrals. I emphasize this is only possible with the Feynman $i\epsilon$ prescription: if we used a retarded or advanced contour, then the poles obstruct us from deforming our contour to the imaginary axis.

Recitation 4-5

Wentao Cui

March 10, 2023

1 The Path-Integral in Quantum Mechanics

1.1 Introduction and Interpretation

Before talking about QFT, we'll discuss the path-integral in QM, which is no more than a $d = 0 + 1$ dimensional QFT. The path-integral formulation of quantum mechanics generalizes the principle of least action. It takes in an action S from a classical theory, and it gives you a quantum theory. This idea is known as path-integral quantization, and alongside canonical quantization it's one of the ways we can bridge the classical and quantum worlds. By and far, I think the path-integral is the most intuitive.

The path-integral can be broken down into 2 fundamental parts.

1. The sole ingredient we feed the path-integral is an action. This is a functional, meaning for each function it returns a number, $S : x(t) \to \mathbb{R}$. Now consider the object $e^{iS[x(t)]/\hbar}$. This is another functional which maps each path to a (now complex) number. The path-integral formulation tells us this is actually a *probability density*, which associates to each path a *weight*.

2. With this probability density, we can compute probabilities. Here one has the formula

$$\langle x_b, t_b | A | x_a, t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} D[x(t)] A e^{iS/\hbar}, \qquad S = \int_{t_a}^{t_b} d\tau L(q, \dot{q})$$

   In particular we can take $A = \mathbb{1}$ to recover what we call the path-integral.

$$Z := \langle x_b, t_b | x_a, t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} D[x(t)] e^{iS/\hbar}$$

   That is, to calculate the probability of starting with an initial state at t_a and ending up in some final state at t_b, we have to sum every single possible path from x_a to x_b, weighted by our density. One may say that the particle takes 'every path', and the weights of different paths differ by a phase. Note that 'all possible paths' includes paths which are not continuous, or those where the particle travels faster than light. This may be eyebrow raising, but is in principle fine–no observables violate causality.

I'd like to emphasize that the crux of the physics lies in point 1, interpreting the object $e^{iS/\hbar}$ as a probability weight. Once you have this, point 2 and everything else is just math, applying the tools of probability theory to do calculations.

You've all seen this idea before, namely in statistical mechanics. In the canonical ensemble one has the Boltzmann weight $e^{-\beta\mathcal{H}}$, and the partition function is akin to the path integral. Again, we have is a probability density associating to each configuration of your system a weight. Both quantum mechanics/field theory and statistical mechanics/field theory are probabilistic approaches towards doing physics, and the very same tools are used in both. Nevertheless, there are 2 major differences.

1. Statistical mechanics takes a deterministic (classical) theory, and uses probabilities to simplify the system so we can compute things. In quantum mechanics probabilities are inherently fundamental to the system.

2. In statistical mechanics, the Boltzmann factor is well-damped for configurations with large energies, and the partition function is convergent. In quantum mechanics the -1 is replaced by an i, which means instead of damping we have phases. The contribution of each path has the same magnitude, meaning the only way to enhance or suppress a configurations are through superposition of phases: constructive and destructive interference.
   Having no exponential suppression is problematic due to convergence issues. However, we may fix this via analytic continuation, by adding a small negative real part to the exponential:

$$Z = \int D[x(t)] \exp\left((1 + i\epsilon) iS/\hbar\right)$$

   This is akin to the Boltzmann weight in statistical physics. It is by no coincidence that I called this $i\epsilon$: it is the same as the $i\epsilon$ in the Feynman Green's function, and this is how it manifests in the path-integral formalism.

1.2 The Classical Limit

We know that quantum mechanics in the limit of $\hbar \to 0$ gives us classical mechanics. Let us see how this happens in the path-integral formalism, where we expect to recover the principle of least action. This will give us intuition of what it means for a system to be 'quantum', as just how 'quantum' it can be. Our starting point is

$$Z = \int D[x(t)] e^{iS/\hbar}$$

As $\hbar \to 0$, the argument of the exponential blows up. The integrand becomes highly oscillatory with respect to the choice of path (the integration variable), and one expects no contribution from most points of the region of integration, where destructive interference is generic. However, there are stationary points where we can have non-trivial contributions.

To see this very explicitly, we consider a toy model that is much simpler. Here the 'path-integral' is a single integral, and we still have an $\hbar$ factor we can toggle.

$$z = \int_{\mathbb{R}} dx e^{if(x)/\hbar}$$

For instance, we can have $f(x) = (x-2)^2$, and consider the integrand as $\hbar \to 0$. [DRAW THIS] In a general small region $x \in [a, a+\delta]$, f will oscillate rapidly, and the integral over the region will vanish. The exception is if f doesn't change much near a point, i.e. $f'(x_0) = 0$. This is called a stationary point, and in this case all contributions of points near x_0 will interfere constructively. Because f is almost constant at x_0, there is enough space for $e^{if/\hbar}$ to contribute before the destructive interference washes it out. We can make this explicit by Taylor expanding near x_0, using that the linear term vanishes:

$$\begin{aligned}
f(x) &= f(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \\
z &= \int dx \exp\left(\frac{i}{\hbar}(f(x_0) + \frac{1}{2}(x-x_0)^2 + \cdots)\right) \\
&= e^{if(x_0)/\hbar} \int dx \exp\left(\frac{i}{2\hbar}(x-x_0)^2 f''(x_0)\right)\left(1 + \frac{i}{\hbar}\frac{1}{3!}(x-x_0)^3 f'''(x_0) + \cdots\right) \\
&= e^{if(x_0)/\hbar}\sqrt{\frac{2\pi i\hbar}{f''(x_0)}}(1 + \text{higher order})
\end{aligned}$$

This is non-zero, and scales inversely with the second derivative at x_0, which measures how quickly $f'(x)$ stops being zero and destructive interference kicks in. In the last step we have performed a Gaussian integral and neglected higher-order terms. This is precisely a saddle-point approximation you might have used in statistical mechanics. If there is more than 1 extremum of the action (can be a minima, maxima, or saddle), one must sum over all these contributions to obtain the path-integral.

In the path-integral where we integrate over paths (infinitely many degrees of freedom), this works the same way. But instead of extremizing f, we are extremizing the action S. In the limit as $\hbar \to 0$ we only pick up contributions from paths $x_c(t)$ where the functional derivative vanishes,

$$\frac{\delta S}{\delta x(t)}_{x_c} = 0$$

Much more on functional derivatives later. This equation says that taking the classical limit, the only contributing paths are those which don't vary the action at first order. This statement is precisely the principle of least action.

With this in mind, we can now interpret $\hbar$. When $\hbar$ is small but non-zero, the dominant trajectories are still those extremizing S, but now we have non-negligible contributions from generic paths. Our perfectly deterministic principle of least action has now been smeared by quantum fluctuations, giving rise to a probabilistic picture. In the opposite limit where $\hbar \to \infty$ the exponential is just 1 for every path: every path interferes constructively with every other, and quantum fluctuations wash out all classical order. A very general way to think about quantum effects are the fluctuations which smear the determinism of classical mechanics, made necessary by all paths contributing to the path-integral. The strength of these fluctuations is controlled by $\hbar$, a constant intrinsic to the theory. This is exactly like in statistical mechanics, where the analogous object to $\hbar$ is $k_B T$. At $T = 0$ your system is in a state which minimizes the energy with probability 1. Finite temperatures induce thermal fluctuations which smear out this order, and as $T \to \infty$ each microstate has the same contribution to the partition funcition: thermal fluctuations wash out all order, and your system has maximal entropy.

2 Gaussian Integrals

In the above, we have used a saddle point approximation to explicitly compute a 'path-integral' with a single variable of integration. Of critical importance was the Gaussian integrand, which allowed us to perform the integral in the first place. Generically in quantum mechanics and QFT, the path-integrals of all free theories can be reduced to computing an infinite number of Gaussian integrals. Even when we include interactions, we Taylor expand the theory around a free theory, which gives us again Gaussian integrals everywhere. Both here and in SFT the role of the Gaussian integral cannot be overemphasized (see: `https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory`), so in preparation we build up the tools for computing them in finite dimensions. In fact, you can think of finite dimensional Gaussian integrals as free $0+0$-dimensional QFTs. These are often called matrix models.

2.1 Gaussian Integrals in 1 Dimension

The Gaussian distribution in 1 dimension is given by the probability density

$$p(\phi \in \mathbb{R}) = \sqrt{\frac{a}{2\pi}} e^{-a\phi^2/2}$$

The nth moment is

$$\langle \phi^n \rangle = \frac{\int d\phi \phi^n e^{-a\phi^2/2}}{Z(a)}, \qquad Z(a) := \int dx e^{-a\phi^2/2} = \sqrt{\frac{2\pi}{a}}$$

One can think of Z as a primitive partition function or path-integral, but all it does now is act as a normalization. In QFT, these moments will correspond to correlation functions. Odd moments vanish because the integral is odd. We can compute even moments by differentiating the normalization integral with respect to a:

$$\langle \phi^{2n} \rangle = \frac{1}{I(a)} \frac{\partial^n}{\partial(-\frac{a}{2})^n} I(a) = \frac{(-2)^n}{I(a)} \frac{\partial^n}{\partial a^n} \int d\phi e^{-a\phi^2/2} = \frac{(2n-1)!!}{a^n}$$

Alternatively we can make use of the generating functional approach, which generalizes nicely to higher (and infinite) dimensions. We introduce the generating functional

$$Z(J) = \int d\phi e^{-a\phi^2/2 + J\phi} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a}, \qquad \langle x^n \rangle = \frac{1}{Z(0)} \frac{\partial^n}{\partial J^n}_{J=0} Z(J)$$

We can explicitly compute J by completing the square (a very important tool when working with path-integrals). In the other equation, each J-derivative brings a power of x down into the integrand: taking n derivatives and setting $J = 0$, we recover the n-th moment.
Now we compute the $2n$th moment using the expression for J. The first J-derivative of Z brings down a power of J/a. Since we set $J = 0$ at the end of the computation, one of the remaining $2n - 1$ derivatives must hit the factor of J. This gives the recursion relation

$$\langle \phi^{2n} \rangle = \frac{2n-1}{a} \langle \phi^{2(n-1)} \rangle \qquad \Rightarrow \qquad \langle \phi^n \rangle = \frac{(n-1)!!}{a}$$

2.2 Gaussian Integrals in n Dimensions

The Gaussian distribution in n dimensions is given by the probability density

$$p(\phi_i \in \mathbb{R}^n) = \left(\frac{\det A}{2\pi} \right)^{n/2} e^{-\frac{1}{2}\phi_i A_{ij} \phi_j}, \qquad A^\intercal = A$$

For instance, if $A = a\mathbb{1}$ we have $p(\phi) = \left(\frac{a}{2\pi}\right)^{n/2} e^{-\frac{1}{2}\phi_i \phi_i}$. To compute moments we define the generating functional

$$Z(J) = \int d^n\phi e^{-\frac{1}{2}\phi_i A_{ij} \phi_j + J_i \phi_i}, \qquad \langle \phi_{i_1} \cdots \phi_{i_k} \rangle = \frac{1}{Z(0)} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_k}}_{J=0} Z(J)$$

We can compute the generating functional by completing the square for matrices:

$$-\frac{1}{2}\phi_i A_{ij} \phi_j + J_i \phi_i = -\frac{1}{2} \left(\phi_i A_{ij} \phi_j - 2 J_i \phi_i \right) = (\phi - A^{-1}J)_i A_{ij} (\phi - A^{-1}J)_j - J_i A^{-1}_{ij} J_j$$

$$Z(J) = \int d^n\phi \exp\left(-\frac{1}{2} \left(\phi_i A_{ij} \phi_j - J_i A^{-1}_{ij} J_j \right) \right) = \frac{(2\pi)^n}{\det A} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

This is a useful trick. Now we can proceed as the 1D case to calculate correlators. The first derivative $\frac{\partial}{\partial J_{i_k}}$ brings down $A_{i_k j} J_j$, using that A is symmetric. One of the remaining derivatives must eliminate the J_j term. Summing over all possibilities one has

$$\langle \phi_{i_1} \cdots \phi_{i_k} \rangle = \sum_{\text{Wick}} \langle \phi_a \phi_b \rangle \cdots \langle \phi_c \phi_d \rangle = \sum_{\text{Wick}} A^{-1}_{ab} \cdots A^{-1}_{cd}$$

The sum is over Wick constractions of the indices $\{i_1, \ldots, i_k\}$. This is the set of all pairs we can form out of the indices. A pair is often denoted by a connector, for instance:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \overbrace{\phi_1 \phi_2} \overbrace{\phi_3 \phi_4} \rangle + \langle \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} \rangle + \langle \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} \rangle = A^{-1}_{12} A^{-1}_{34} + A^{-1}_{13} A^{-1}_{24} + A^{-1}_{14} A^{-1}_{23}$$

For instance, if our Gaussian has $A = a\mathbb{1}$, i.e. $p(x) \propto e^{-\frac{1}{2} x_i x_i}$, then

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = a(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23})$$

Recitation 5

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1 Path Integrals in QFT

1.1 Bridge: to Infinite Degrees of Freedom

As we've discussed, the path-integral formulation of QFT is tantamount to Gaussian integrals with a infinite number of degrees of freedom. It is thus important to discuss how calculus and linear algebra change when we go from a finite number of degrees of freedom to an infinite number, often in a continuous domain. In this section we establish a dictionary between them.

Degrees of freedom:

- Finite vector $\to$ continuous function, i.e. $\phi_i \to \phi(x)$
  When working with Gaussians, we had degrees of freedom ϕ_i. For i continuous, like a space or a time, this becomes a function $\phi(x)$. For our QFT purposes this will be a field depending on space and time. Often we might want both continuous and discrete labels, such as a vector field $A_i(x)$. One can also view QM as a $0+1$-dimensional QFT. Here the degree of freedom is time, and the 'fields' are the positions X_i.

Basic operations:

- Sums $\to$ integrals, i.e. $\sum_i \to \int dx$
- Functions $\to$ functionals, i.e. $f(\phi_i) \to F[\phi(x)]$
  The functionals of our interest include things like the action or Lagrangian. Generally they will be an integral of some function of $\phi(x)$, and can also be functions of other points x':

$$F[\phi(x)] = \int dx f(\phi(x)), \qquad H_{x'}[\phi(x)] = \int_{\mathbb{R}} dx h(x, x')\phi(x)$$

Calculus:

- (Partial) derivatives $\to$ functional derivatives, i.e. $\dfrac{\partial}{\partial \phi_i} \to \dfrac{\delta}{\delta \phi(x)}$
  A normal derivative measures how much a function changes with respect to a finite degree of freedom. A functional derivative measures how much a functional changes with respect to a function (infinite degree of freedom). This can be made explicit for some $F[\phi(x)]$ by perturbing by some test function $\eta(x)$, and seeing how much F varies:

$$\Delta F[\phi(x), \eta(x)] = \lim_{\epsilon \to 0} \frac{F[\phi(x) + \epsilon\eta(x)] - F[\phi(x)]}{\epsilon} = \int dx \eta(x) \frac{\delta F}{\delta \phi(x)}$$

  It is instructive to compare this to a directional derivative, which we have for discrete ϕ_i, where we have some vector-perturbation n_i.

$$\Delta f(x_i, n_i) = \lim_{\epsilon \to 0} \frac{f(x_i + \epsilon n_i) - f(x_i)}{\epsilon} = \sum_i n_i \frac{\partial f}{\partial x_i}$$

We recover the partial derivative in the j-direction for $n_i = \delta_{ij}$, and similarly we recover the functional derivative in the x_0-th direction for $\eta(x) = \delta(x - x_0)$. This measures the change in F when we add a small perturbation at x_0. [DRAW FIGURE]

- Integrals $\to$ path integrals, i.e. $\int \prod_i d\phi_i \to \int D\phi(x)$
  This is more intuitive. One has infinite degrees of freedom, so we have to integrate over each. One can think of integrating over the values of a field each each point in spacetime.

Linear algebra:

- Matrices $\to$ operators, i.e. $A_{ij} \to K(x, x')$
  This should be familiar from quantum mechanics. A matrix is just a finite dimensional operator. In the continuous case the vector space on which the operator acts is infinite-dimensional and each index becomes a continuous variable.
  The identity operator $\mathbb{1}_{ij}$ becomes the Dirac-delta, $\delta(x-x')$. We can also have differential operators like ∂_x, which don't have a relevant finite analog (it would look something like $\Delta\phi_i = \phi_i - \phi_{i-1}$, which generically does not have any meaning).

- Matrix products: $\to$ operator products, i.e. $A_{ij}B_{jk}\phi_k \to \int dx'dx''A(x, x')B(x', x'')\phi(x'')$
  Index sums become integrals. We can confirm that the Dirac-delta is the identity, since

$$\int dx' A(x, x')\delta(x - x'') = A(x, x'')$$

- Matrix inverses $\to$ Green's functions, i.e. $A^{-1}_{ij} \to G_K(x, x')$
  The matrix inverse is defined as the operator that, when multiplied with the original operator, yields the identity:

$$\sum_j A_{ij}A^{-1}_{jk} = \delta_{ik}$$

  Using the above we take the continuous analog:

$$\int dx' K(x, x')K^{-1}(x', x'') = \delta(x - x'')$$

  We recognize this as the Green's function for the operator K. For instance for the Klein-Gordon operator we have

$$\int dx'\delta^{(4)}(x - x')(-\partial^2_{x'} + m^2)G(x', x'') = \delta^{(4)}(x - x'') \quad \iff \quad (-\partial^2_x + m^2)G(x, x'') = \delta^{(4)}(x - x'')$$

- Eigendecompositions:
  These are defined in the same way

$$\sum_i A_{ij}v_j - \lambda v_j \to \int dx' K(x, x')f(x') = \lambda f(x)$$

  While the eigenvalues stay numbers, eigenvectors now become functions. This should be familiar from quantum mechanics: the eigenvectors of the derivative operator $K(x, x') = \delta(x - x')\frac{d}{dx}$ are just the exponential functions $e^{\lambda x}$ with eigenvalues $\lambda \in \mathbb{R}$:

$$\int dx'\delta(x - x')\frac{d}{dx}e^{\lambda x'} = \frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

- Determinants and traces:
  These are defined as usual: as the sum/product of the operator's eigenvalues. In the continuous case, we will have an infinite sum (i.e. integral) or product. Often these will be infinite, but taking ratios of determinants or traces of operators gives something finite.

1.2 Path-Integrals for Free Theories

Note: for this section I would recommend opening the notes from the previous recitation and compare this to what I'll be doing today. You will see that the procedure is almost identical.

1.2.1 The Distribution

All the hard work is behind us. We can use the same techniques we had for computing correlators of finite Gaussian integrals to path-integrals. In particular, the Wick contractions we saw last week will give rise to Feynman diagrams.

The path-integral for a field theory in n dimensions is based on the following distribution:

$$p[\phi(x)] = \frac{1}{Z_0} e^{iS[\phi]}, \qquad \phi \in \mathbb{R}^{1,n-1}$$

where we define the partition function Z_0, for now thought of as a normalization constant

$$\mathcal{Z}_0 = \int D[\phi(x)] e^{iS[\phi(x)]}$$

By free field theory, we mean that the action is quadratic:

$$\begin{aligned} iS &= \int d^4x \left[\frac{i}{2}(-\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)\right] = \frac{1}{2}\int d^4x \phi(\partial^2 - m^2)\phi \\ &= \frac{i}{2}\int d^4x d^4y\, \phi(y)\left[\delta^{(4)}(x-y)(\partial_x^2 - m^2)\right]\phi(x) \\ &= \frac{i}{2}\int đ^4p đ^4q\, \phi(q)\left[(2\pi)^4\delta^{(4)}(p-q)(-p^2-m^2)\right]\phi(p) \\ &= -\frac{1}{2}\int đ^4p đ^4q\, \phi(q)K(q,p)\phi(p), \qquad\qquad K(q,p) = i(2\pi)^4\delta^{(4)}(p-q)(p^2+m^2) \end{aligned}$$

In the last line I write this in momentum space. This looks almost identical to a finite Gaussian integral, where instead of $\phi_i A_{ij}\phi_j$ we have $\phi(p)K(p,q)\phi(q)$. In momentum space K is particularly simple, with no derivatives. Note that the $\delta^{(4)}(p-q)$ makes this operator diagonal: we have no mixing between $\phi(p)$ and $\phi(q)$ unless $p = q$. If you made me write this as a finite dimensional operator, it would morally be

$$A_{jk} = i(k^2+m^2)\delta_{jk} = i\,\mathrm{diag}(\cdots, 4+m^2, 1+m^2, m^2, 1+m^2, 4+m^2, \cdots)$$

This way of viewing makes the Green's function, or matrix inverse, particularly clear. Proceeding with the matrix analogy, the inverse is

$$A^{-1}_{jk} = \frac{-i}{k^2+m^2}\delta_{jk} = \mathrm{diag}\left(\cdots, \frac{-i}{4+m^2}, \frac{-i}{1+m^2}, \frac{-i}{m^2}, \frac{-i}{1+m^2}, \frac{-i}{4+m^2}, \cdots\right)$$

We thus can immediately write down the Green's function,

$$G(q,p) = \delta(p-q)\frac{-i}{p^2+m^2}$$

1.2.2 Correlators

We seek to compute correlation functions of operators with respect to this 'probability distribution':

$$G_n := \langle 0|\mathrm{T}\hat{\phi}(x_n)\cdots\hat{\phi}(x_1)|0\rangle = \frac{1}{Z_0}\int D\phi e^{iS[\phi]}\phi(x_n)\cdots\phi(x_1)$$

Note that on the left, $\hat{\phi}$'s are operators acting on a Hilbert space, while on the right the ϕ's are functions. These correlators give us information about fluctuations of the field around the vacuum state.

To compute these objects we introduce the generating functional

$$\begin{aligned} Z[J] &= \int D\phi \, \exp\left[\frac{i}{2}\int đ^4p đ^4q \, \phi(q)K(q,p)\phi(p) + i\int đ^4p \, J(p)\phi(p)\right] \\ &= \frac{C}{\sqrt{\det K}} \exp\left[\frac{i}{2}\int đ^4p đ^4q J(q)K^{-1}(q,p)J(p)\right] \end{aligned}$$

In the last line we have completed the square to perform the functional integral. This technique is almost identical to what we did for matrices. The factor of C is the product of all the 2π factors which follow from the Gaussian integral.

From last week's discussion we can immediately write down the formula for computing correlators:

$$\begin{aligned} \langle\phi(p_n)\cdots\phi(p_1)\rangle &= \frac{1}{Z(0)}\left(-i\frac{\partial}{\partial J(p_n)}\right)\cdots\left(-i\frac{\partial}{\partial J(p_1)}\right)_{J=0} Z(J) \\ &= \sum_{\text{Wick}} K^{-1}(p_a,p_b)\cdots K^{-1}(p_c,p_d) \end{aligned}$$

In the first line we have obligatory factors of $-i$ for each derivative due to our convention to have an i in front of $J(p)$ in the integral. In the last line, we sum over all Wick contractions of $\{p_1, p_2, \ldots, p_n\}$, best thought of as indices in parallel with the matrix case. If at any point you get confused about this, I suggest rephrasing your question in terms of Gaussian integrals. Note that here I work in momentum space, where K^{-1} is particularly simple, acting without derivatives. This is what is conventionally done in QFT. If you insisted I work in position space, the same process goes through, the only difference being the form of $K^{-1}(y,x)$ compared to $K^{-1}(q,p)$. To go back to position space, you can also just take the Fourier transform of the correlation function with respect to each of the momenta.

Most important is the 2-point function, for which there is a single Wick contraction to consider. We collect all our results:

$$\begin{aligned} K^{-1}(q,p) = G_F(q,p) &= \langle 0|\mathrm{T}\phi(q)\phi(p)|0\rangle = \frac{-i}{p^2+m^2-i\epsilon}\delta(p-q) \\ K^{-1}(x,y) = G_F(x,y) &= \langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle = \int đ^4k \frac{-i}{k^2+m^2-i\epsilon}e^{ik\cdot(x-y)} \end{aligned}$$

That is, the inverse of the kinetic term, the Feynman Green's function, and the 2-point correlator are all identical. We call this object the propagator.

Example: let us compute the 4-point function of a free scalar theory. We can copy the Wick contractions almost verbatim from last week's example:

$$\begin{aligned} \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \langle\overbrace{\phi(x_1)\phi(x_2)}\overbrace{\phi(x_3)\phi(x_4)}\rangle + \langle\overbrace{\phi(x_1)\overbrace{\phi(x_2)\phi(x_3)}\phi(x_4)}\rangle + \langle\overbrace{\phi(x_1)\overbrace{\phi(x_2)\phi(x_3)}\phi(x_4)}\rangle \\ &= K^{-1}(x_1,x_2)K^{-1}(x_3,x_4) + K^{-1}(x_1,x_3)K^{-1}(x_2,x_4) + K^{-1}(x_1,x_4)K^{-1}(x_2,x_3) \end{aligned}$$

[DRAW DIAGRAMMATICALLY]

2 Interactions and The Feynman Calculus

2.1 Adding Interactions

Everything we have done is for a free theory, where we have only a quadratic term in our field ϕ. Free theories lead to Gaussian path-integrals, which is one of the very few cases where we can perform the

path-integral explicitly. We saw this above. Adding interactions will lead to 2 major difficulties:

- More complicated Lagrangians. Generically, an action will have cubic and higher-order terms, such as a ϕ^4 theory:

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 + \lambda \phi^4$$

  For generic cases like this, we can only compute the path-integral perturbatively, expanding around a free theory assuming small λ. The convergence of this series is a tricky business.

- A different vacuum. We denote the vacuum of a free theory by $|0\rangle$, and the interacting theory by $|\Omega\rangle$. These are different states: adding terms to the Lagrangian changes the vacuum. Since free theories are exactly solvable, we understand $|0\rangle$ much better than $|\Omega\rangle$. Perturbation theory will allow us to understand fluctuations about $|\Omega\rangle$ in terms of those about $|0\rangle$.

Now we turn to the evaluation of the correlator of an interacting theory:

$$G_n := \langle \Omega | \mathrm{T} \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$$

We will do this in perturbation theory, in principle to an arbitrary order. Again the key ingredient will be Wick's theorem. Nothing about what we do is unique to field theories: the same process holds for matrix theories, if we add higher order terms to our Gaussian integrals.

To do this, we split our action into a free and interaction piece:

$$S = S_0 + S_I, \qquad S_0 = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

From class, we have the central 'magic formula' of Gell-Mann and Low which relates correlators about the interaction vacuum to those about the free vacuum:

$$G_n = \frac{\langle 0 | \mathrm{T} \phi(x_1) \cdots \phi(x_n) e^{iS_I} | 0 \rangle}{\langle 0 | \mathrm{T} e^{iS_I} | 0 \rangle}, \qquad S_I = -\int dt H_I$$

Perturbation theory, then, is the process of expanding the e^{iS_I} exponential to a given order and computing correlators. By now, you know 2 methods to compute these correlators, which give us 2 ways of doing perturbation theory. They amount to doing the same thing, which are Wick contractions.

1. Hamiltonian framework: using the free-field expansions of ϕ into creation and annihilation operators
2. Path-integral framework: view these as correlators of a Gaussian density, and perform the Gaussian integrals explicitly.

This formula contains all the physics, and is something all of you should remember. The rest is just combinatorics, and will be handled in the next recitation.

Recitation 6

Wentao Cui

March 24, 2023

1 Combinatorics

We consider the 3-point interaction $\mathcal{L}_I = \frac{\lambda}{3!}\phi^3$, and compute to second order in λ the object:

$$G_2 = \langle\Omega|\mathrm{T}\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle}{\langle 0|\mathrm{T}e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle} = \frac{N}{D}$$

We will compute this object using 3 different approaches to simplify the combinatorics, each simpler and more useful than the last.

1.1 Wick Contractions

We start with the denominator.

$$\begin{aligned}\langle 0|\mathrm{T}e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle &= \sum_{s=0}^{2}\frac{(i\lambda/3!)^s}{s!}\int d^4z_1\cdots d^4z_s\langle 0|\phi^3(z_1)\cdots\phi^3(z_s)|0\rangle + \mathcal{O}(\lambda^3)\\ &= 1 + i\lambda\int d^4z\langle 0|\phi^3(z)|0\rangle - \frac{\lambda^2}{2(3!)^2}\int d^4z_1d^4z_2\langle 0|\phi^3(z_1)\phi^3(z_2)|0\rangle\end{aligned}$$

The order λ^1 term vanishes, since it is a correlator of an odd number of fields, so there are no ways to Wick-pair them. It remains to evaluate the object $\langle 0|\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)|0\rangle$. There are two distinct ways of Wick contracting, which we summarize in a table:

Contraction Type	Ways to Contract	Value (per contraction)	Diagram
$\langle\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\rangle$	3×3	$G_F(z_1,z_1)G_F(z_1,z_2)G_F(z_2,z_2)$	
$\langle\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\rangle$	$3\times 2\times 1$	$G_F(z_1,z_2)^3$	

Putting everything together, we have

$$\begin{aligned}D &= 1 - \lambda^2\int d^4z_1d^4z_2\left[\frac{1}{8}G_F(0,0)^2G_F(z_1,z_2) + \frac{1}{12}G_F(z_1,z_2)^3\right] + \mathcal{O}(\lambda^3)\\ \frac{1}{D} &= 1 + \lambda^2\int d^4z_1d^4z_2\left[\frac{1}{8}G_F(0,0)^2G_F(z_1,z_2) + \frac{1}{12}G_F(z_1,z_2)^3\right] + \mathcal{O}(\lambda^3)\end{aligned}$$

Now the numerator.

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle = \sum_{s=0}^{2}\frac{(i\lambda/3!)^s}{s!}\int d^4z_1\cdots d^4z_s\langle 0|\phi(x_1)\phi(x_2)\phi^3(z_1)\cdots\phi^3(z_s)|0\rangle + \mathcal{O}(\lambda^3)$$
$$= \langle 0|\phi(x_1)\phi(x_2)|0\rangle + i\lambda\int d^4z\langle 0|\phi(x_1)\phi(x_2)\phi^3(z)|0\rangle$$
$$- \frac{\lambda^2}{2(3!)^2}\int d^4z_1d^4z_2\langle 0|\phi(x_1)\phi(x_2)\phi^3(z_1)\phi^3(z_2)|0\rangle$$

The first term is the result from the free theory, in the absence of interactions. The linear term in λ is again odd in ϕ's, so vanishes. We need to compute the quadratic term. There are several ways of Wick contracting, given below.

Contraction Type	# Ways	Value (per contraction)	Diagram	Notes
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 2\times 3$	$G^F_{x_1,z_1}G^F_{x_2,z_1}G^F_{z_1,z_2}G^F_{z_2,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 2\times 3$	$G^F_{x_1,z_2}G^F_{x_2,z_2}G^F_{z_1,z_1}G^F_{z_1,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,z_1}G^F_{x_2,z_2}G^F_{z_1,z_1}G^F_{z_2,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,z_2}G^F_{x_2,z_1}G^F_{z_1,z_1}G^F_{z_2,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 3\times 2$	$G^F_{x_1,z_1}G^F_{x_2,z_2}G^F_{z_1,z_2}G^F_{z_1,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 3\times 2$	$G^F_{x_1,z_2}G^F_{x_2,z_1}G^F_{z_1,z_2}G^F_{z_1,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,x_2}G^F_{z_1,z_1}G^F_{z_1,z_2}G^F_{z_2,z_2}$		vacuum
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×2	$G^F_{x_1,x_2}G^F_{z_1,z_2}G^F_{z_1,z_2}G^F_{z_1,z_2}$		vacuum

Putting everything together, we have

$$\frac{N}{D} = \frac{G_F(x_1,x_2)(1+\text{vac.}) + N_{\text{no vac.}}}{1+\text{vac.}} + \mathcal{O}(\lambda^3) = G_F(x_1,x_2) + N_{\text{no vac.}} + \mathcal{O}(\lambda^3)$$
$$= G_F(x_1,x_2) - \frac{\lambda^2}{2}\int d^4z_1d^4z_2[G_{x_1,z_1}G_{x_2,z_1}G_{z_1,z_2}G_{0,0} + \frac{1}{2}G_{x_1,z_1}G_{x_2,z_2}G^2_{0,0} +$$
$$+ G_{x_1,z_1}G_{x_2,z_2}G_{z_1,z_2}G_{z_1,z_2}]$$

1.2 Feynman Rules in Position Space

The Feynman rules give a prescription to evaluate the Gell-Mann Low formula, which takes less thinking than Wick contractions, although the combinatorics (read: symmetry factors) can be more confusing. The idea is to write down a set of diagrams corresponding to the terms in the Taylor expansion of e^{iS_I}. We can compute each diagram using the Feynman rules, and adding a combinatorial factor.

The procedure is as follows.

1. Draw the external and internal vertices to desired order in perturbation theory.

   - An external vertex has 1 leg, and correspond to the position (or momenta) of fields in the correlator. For a given correlator, the number of external vertices is always the same.
   - An internal vertex corresponding to $S_I = \lambda\phi^k/k!$ has k legs. Each external vertex corresponds to a position (or momentum) z which is integrated over, coming from a power of $S_I = \int d^d z(\cdots)$ when Taylor expanded. To compute up to λ^r in perturbation theory, one must consider all diagrams with $\leq r$ internal vertices.

2. Connect the vertices using the legs in all (topologically distinct) ways to get all of the diagrams. Drop diagrams which have vacuum bubbles (i.e. a disconnected component with no external legs attached). These are cancelled out by the denominator in the Gell-Mann Low formula (see: your problem set.)

3. Feynman rules. For each diagram, write its contribution using the following 'Feynman rules'.

   - Each external vertex contributes as 1. Sometimes you may see this written with a factor $e^{-ix\cdot p}$: this is just a Fourier factor to convert to momentum space.
   - Each internal vertex corresponding to a term $i\lambda\phi^k/k! \subset iS_I$ contributes as $i\lambda$.
   - Each edge between vertices corresponds to a Wick contraction between the fields at those points, hence contributes $G_F(x-y)$.

4. Symmetry factors. This is the most confusing step.
   Note that the Feynman rules in (3) don't include any $k!$ or $r!$ (perturbation order) combinatorial factors. This is because we have assumed naïve combinatorics, amounting to 2 statements:

   (a) There are $k!$ ways to permute the legs of each vertex, each of which prescribe a way to 'connect' the vertex to the rest of the diagram via Wick contractions. If each permutation leads to a distinct contraction, we pick up a $k!$ factor which cancels out with the $1/k!$ factor in associated to the vertex in S_I. Hence, we can ignore both.

   (b) Working to order r in perturbation theory, there are $r!$ ways to permute the different internal vertices. If each permutation leads to a distinct set of contractions, we pick up an $r!$ factor which cancels out with the $1/r!$ in the Taylor expansion of the exponential. Hence, we can ignore both.

   Corrected combinatorics: there are certain situations where this naïve counting breaks down. In particular, some permutation of vertex legs or internal vertices lead to the same Wick contraction, so the naïve prescription is overcounting. To correct this we divide by a 'symmetry factor', which should be assigned (multiplicatively) in the following cases:

   (a) Propagators starting and ending at the same point: $\times\frac{1}{2}$
   Since these are Wick contracted, permuting them leads to the same Wick contraction.

(b) j propagators connecting 2 internal vertices: $\times \frac{1}{j!}$
Naïvely we pick up a factor of $j!$ from permuting these legs of each vertex, so $j!^2$ in total. However, there are only $j!$ ways to Wick contract, hence we overcount by $j!$.

(c) j internal vertices which are interchangeable in how they connect to the diagram: $\times \frac{1}{j!}$
Permuting these vertices lead to identical Wick contractions, so we naïvely overcount by $j!$

Let us use these to compute the example from above. We have the following diagrams and their contributions. Note that $G_F(z,z) = G_F(0,0)$ by translation symmetry.

Diagram	Symm. Factor	Value
	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2 \int d^4z_1 d^4z_2 G_F(x_1,z_1) G_F^2(z_1,z_2) G_F(x_2,z_2)$
	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2 G_F(0,0) \int d^4z_1 d^4z_2 G_F(x_1,z_1) G_F(x_2,z_1) G_F(z_1,z_2)$
	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{4}(i\lambda)^2 G_F^2(0,0) \int d^4z_1 G_F(x_1,z_1) \int d^4z_2 G_F(x_2,z_2)$
(vac.)	–	N/A

Together with the 0th order piece, we have our result from before, using the shorthand $G_{x,y} := G_F(x,y)$.

$$G_2 = G_{x_1,x_2} - \frac{\lambda^2}{2} \int d^4z_1 d^4z_2 [G_{x_1,z_1} G_{x_2,z_1} G_{z_1,z_2} G_{0,0} + \frac{1}{2} G_{x_1,z_1} G_{x_2,z_2} G_{0,0}^2 + G_{x_1,z_1} G_{x_2,z_2} G_{z_1,z_2} G_{z_1,z_2}]$$

1.3 Feynman Rules in Momentum Space

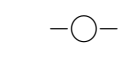
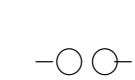
These are very similar to the Feynman rules in position space. However, the diagrams are simpler to evaluate because we can make use of momentum conservation and the explicit form of the propagator in momentum space to simplify our expression.

The procedure is as follows.

1. Draw the external and internal vertices to desired order in perturbation theory. (same)
2. Draw down all topologically distinct diagrams without vacuum bubbles. (same)
3. Feynman rules: these are now different.
   - Each external vertex contributes as 1. Sometimes you may see this written with a factor $e^{ix\cdot p}$: this is just a Fourier factor to convert to position space.
   - Each internal vertex contributes $i\lambda\delta^{(4)}(\sum p_{\text{in}} - \sum p_{\text{out}})$.
     - Note: we can see how this comes from taking the position space Feynman rules: we have a Fourier phase for each fields at an internal vertex point x, totalling to $e^{-ix(\sum iq_i)}$. Integrating over the internal vertex location x produces the desired δ-function.

     We account for the δ-function by imposing momentum conservation at each vertex. This fixes all internal momenta, aside from those running in a loop, which are still integrated over. This includes the total momentum conservation $(2\pi)^4\delta^{(4)}(\sum p_{\text{in}} - \sum p_{\text{out}})$
   - Each edge (Wick contraction) contributes $G_F(p) = \dfrac{-i}{p^2 + m^2 - i\epsilon}$
4. Symmetry factors: these are the same as in position space. (same)

Let us use these to compute the example from above. We have the following diagrams and their contributions:

Diagram	Symm.	Value
	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \int \frac{d^4k(-i)^2}{(k^2+m^2-i\epsilon)((p-k)^2+m^2-i\epsilon)}$
	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \frac{-i}{(p_1-p_2)^2+m^2-i\epsilon}\int \frac{d^4k(-i)}{k^2+m^2-i\epsilon}$
	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{4}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \left(\int \frac{d^4k(-i)}{k^2+m^2-i\epsilon}\right)^2$
(vac.)	–	N/A

Summing the contributions, we get

$$G_2(p_1,p_2) = -\frac{\lambda^2}{2}(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{1}{p_1^2+m^2-i\epsilon}\frac{1}{p_2^2+m^2-i\epsilon}$$
$$\left[\int \frac{d^4k}{(k^2+m^2-i\epsilon)((p-k)^2+m^2-i\epsilon)} + \frac{1}{(p_1-p_2)^2+m^2-i\epsilon}\int \frac{d^4k}{k^2+m^2-i\epsilon} + \frac{1}{2}\left(\int \frac{d^4k}{k^2+m^2-i\epsilon}\right)^2\right]$$

Recitation 7

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1 Representations of the Lorentz Group

The Lorentz Group

The Lorentz algebra $\mathfrak{so}(1,3)$ is generated by the set of infinitesimal boosts and rotations. Equivalently, it can be defined by the commutation relations obeyed by the generators $(\mathcal{J}^{\lambda\rho})^{\mu}{}_{\nu}$:

$$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\rho\sigma}] = i(\eta^{\alpha\sigma}\mathcal{J}^{\beta\rho} - \eta^{\alpha\rho}\mathcal{J}^{\beta\sigma} - \eta^{\beta\sigma}\mathcal{J}^{\alpha\rho} + \eta^{\beta\rho}\mathcal{J}^{\alpha\sigma})$$

To get an element of the Lorentz group, we can exponentiate some linear combination of these generators.

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right)$$

The Lorentz algebra is an example of what we call a Lie algebra, defined by some generators, their linear combinations, and a commutator or 'Lie bracket'. The Lorentz group is the exponential of the algebra, or equivalently the algebra is the set of infinitesimal elements of the group.

Representations

A representation of the Lorentz algebra is a map from $\mathfrak{so}(1,3)$ into matrices in a way that preserves the central commutation relation above. It provides an explicit realization of the algebra into a set of matrices that we can manipulate. More formally, a representation is a map $d : \mathfrak{so}(1,3) \to \mathfrak{gl}(V)$ such that

$$[d(\mathcal{J}^{\alpha\beta}), d(\mathcal{J}^{\rho\sigma})] = d([\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\rho\sigma}])$$

Here $\mathfrak{gl}(V)$ is the space of linear transformations of a vector space V, itself equivalent to the space of matrices acting on V. To give a representation then, is to prescribe the vector space V along with the action of the generators (i.e. their matrix forms). Sometimes their matrix forms are clear from context, so we denote a representation by just V.

Any representation of the algebra $\mathfrak{so}(3,1)$ induces a representation $D : \mathrm{SO}(1,3) \to \mathrm{GL}(V)$ of the Lorentz group by exponentiating:

$$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\rho\sigma}d(\mathcal{J}^{\rho\sigma})\right), \qquad D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$$

Here $\mathrm{GL}(n)$ is the space of invertible matrices. The proof of the second equation amounts to the Baker-Campbell-Hausdorff formula.

Field Representations

This holds for general representations. Now we can talk about the representations formed by the set of some fields $\{\Phi_a(x)\} = V$. The action of the Lorentz group is given by

$$D(\Lambda)^{ab}\Psi^b(\Lambda^{-1}x)$$

Another way of saying this is that this is how our field transforms under a Lorentz transformation. The transformation of the spacetime location is given by the inverse of the transformation, which should be familiar from working with scalar fields. We also transform the internal indices, which depend on the representation that the field lives in: examples include scalars, vectors, and tensors, which are given by

$$\phi(x) \to \phi(\Lambda^{-1}x), \qquad A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x), \qquad T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}(\Lambda^{-1}x)$$

The scalar has 1 component, the vector has 4 components, and the tensor has 16. For this reason, we often denote these representations by **1**, **4**, **16**. Comparing this to the general equation, we have $D_{\mathbf{1}}(\Lambda) = 1$, $D_{\mathbf{4}}(\Lambda) = \Lambda^\mu{}_\nu$, $D_{\mathbf{16}}(\Lambda) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta$.

Irreducible Representations

Not all representations are created equal: some are simpler than others, and serve as 'building blocks' from which we can construct all the representations we care about. To talk about these, we first introduce the notion of an invariant subspace W, or ideal, of a representation V as one that is mapped to itself under all Lorentz transformations. That is,

$$D(\Lambda)W = \{D(\Lambda)w, w \in W\} \subseteq W$$

If $W \subseteq V$ is invariant, it's not hard to show that $W^\perp$ is also invariant, and thus one can decompose V into a direct sum:

$$V = W \oplus W^\perp$$

If each subspace is invariant, the action of the Lorentz group acts independently on each subspace, with no mixing. Viewing the $D(\Lambda)$'s as matrices acting on V, this means that we can block diagonalize all the matrices at once, as off-diagonal terms correspond to mixing between W and $W^\perp$. Obviously V and $\emptyset$ are always invariant, these are the trivial invariant subspaces of V.

A representation is irreducible (irrep.) if there are no non-trivial invariant subspaces. That is, we cannot block diagonalize all $D(\Lambda)$'s at once, meaning one cannot factor V into smaller building block representations. In this sense, you may think of irreps as prime numbers. Let us see whether our familiar scalar, vector, and 2-tensor field representations are irreducible.

1. Scalars. All Lorentz transformations act trivially as the identity on a dimension 1 object, the matrix is just (1). There are no non-trivial subspaces, so **1** is an irrep.

2. Vectors. This is the representation **4**. It is irreducible, as one cannot simultaneously block diagonalize the rotations and boosts. It is called the fundamental representation.

3. Tensors. This is the representation **16**. It has 3 invariant subspaces.
   1: The scalar subspace, given by span$\{\eta^{\mu\nu}\}$; this is left invariant under Lorentz by definition.
   $$(D_{\mathbf{16}}(\Lambda)\eta)^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$$
   6: The space of antisymmetric tensors of dimension 6, $B^{\mu\nu} = -B^{\nu\mu}$. One can check:
   $$(D_{\mathbf{16}}(\Lambda)B)^{\mu\nu} := \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\alpha\beta} = -\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\beta\alpha} = -(D_{\mathbf{16}}(\Lambda)B)^{\nu\mu}$$
   It turns out that the **6** decomposes into 2 invariant dimension 3 subspaces, $\mathbf{6} = \mathbf{3} \oplus \bar{\mathbf{3}}$
   Here the **3** and $\bar{\mathbf{3}}$ are self-dual and anti-self-dual tensors, i.e. $G^{\mu\nu} = \pm i\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$.
   9: The space of symmetric tensors of dimension 10, $S^{\mu\nu} = S^{\nu\mu}$.
   In total, the decomposition of the tensor representation into irreps is thus
   $$\mathbf{16} = \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{9}$$

Our goal is to classify all the irreducible representations of the Lorentz group. once we do this, we can build any other representation we care about.

2 Interlude: SU(2)

Before doing this for the Lorentz group, let us first turn to a simpler example which you should all be familiar with, SU(2). This is the Lie group of 2×2 unitary matrices with unit determinant,

$$\mathrm{SU}(2) = \{U \in M_{2\times 2}(\mathbb{C}), \quad U^\dagger U = UU^\dagger = 1, \quad \det U = 1\}$$

The corresponding Lie algebra is $\mathfrak{su}(2)$. It has 3 generators, $\{J_1, J_2, J_3\}$ satisfying the relation

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

Exponentiating this algebra gives SU(2): any group element can be written as $U = e^{i\theta_i J_i}$. It's often easier to look for representations of the algebra instead of the group, so we will look for all irreducible representations of $\mathfrak{su}(2)$. Exponentiating these, we can recover all the irreducible representations of SU(2). That is, we look for a set of $r \times r$ matrices $d_s(J_i)$ satisfying the above commutation relation, that cannot be simultaneously block diagonalized.

The classification is simple. There is a unique $\mathfrak{su}(2)$ irrep for each dimension $r \geq 1$. These correspond to particles of different spins, which is made clear by defining $l = (r-1)/2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. In this context $r = 2l + 1$ is the number of orthogonal states a spin l particle can have, often labelled $m = -l, \ldots l$.

What are the $r \times r$ matrices that satisfy our commutation relation? For spin-0 the matrix is trivial, $d_0(J_1, J_2, J_3) = (0)$: angular momentum acts trivially. For spin-1/2, $r = 2$, and the matrices are given by the familiar Paulis. The objects on which these act form a vector space, and are called Pauli spinors.

$$d_{1/2}(J_1, J_2, J_3) = \frac{1}{2}(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0, & -1 \end{pmatrix}\right)$$

In $d \geq 3$, we have generalizations of the Pauli matrices. These are called Wigner D-matrices, and are given by

$$d_l(J_1) \propto \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad d_l(J_2) \propto \begin{pmatrix} 0 & -i & & \\ i & 0 & -i & \\ & i & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad d_l(J_3) = \mathrm{diag}(-l, -1+1, \ldots, l-1, l)$$

To summarize: all irreducible representations of $\mathfrak{su}(2)$ are classified by an integer $l \geq 1$ representing its dimension. Exponentiating them gives representations of SU(2).

3 Classification of Irreducible Representations of the Lorentz Group

3.1 The Classification

Now we've built up enough foundations to perform our classification. This hinges on one fact which we will show–the Lorentz algebra looks like 2 copies of $\mathfrak{su}(2)$.

We often think of the Lorentz algebra $\mathfrak{so}(1,3)$ being generated by the rotations and boosts J_i and K_i. We will work in an alternative basis $\{J_i^\pm\}$, given by the generators

$$J_i^\pm := \frac{1}{2}(J_i \pm iK_i)$$

Using the commutation relations of rotations and boosts, we can compute the commutators in this basis:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm}, \qquad [J_i^{\pm}, J_j^{\mp}] = 0$$

This form is particularly insightful, and should be familiar. Namely, we have 2 copies $\{J_i^+\}$ and $\{J_i^-\}$ of the $\mathfrak{su}(2)$ algebra, which are completely decoupled. We have thus shown the result

$$\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

How do we classify irreps of a direct product? Generally, we may have something like $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where we know the irreps of $\mathfrak{h}_{1,2}$ as $\{d_a^1, V_a\}$ and $\{d_b^2, W_b\}$. The irreps of the larger algebra is given by the tensor product: $\{d_a^1 \otimes d_b^2, V_a \otimes W_b\}$. Practically, this means that we take the tensor product of the vector spaces on which they act, and the tensor product of the matrices as well. The irreps of $\mathfrak{g}$ are then labelled by the pair (a, b).

Let us return to the Lorentz algebra. We know the irreps of $\mathfrak{su}(2)$ are labelled by a non-negative half-integer. The irreps of $\mathfrak{so}(1,3)$ then, are labelled by a pair (j_+, j_-), for $j_+, j_- \in \{0, \frac{1}{2}, 1, \ldots\}$. By the usual rules of the vector space tensor product, the dimension is given by the product of the j_+ and j_- dimensions:

$$\dim(d_{j_+,j_-}) = (2j_+ + 1)(2j_- + 1)$$

In this notation, we may now write the Lorentz scalar as $\mathbf{1} = (0,0)$, the vector as $\mathbf{4} = (\frac{1}{2}, \frac{1}{2})$, and the tensor as $\mathbf{16} = \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{9} = (0,0) + (0,1) + (1,0) + (1,1)$.

If $j_+ + j_-$ is an integer we call this a tensor representation, and if it a half-integer we call it a spinor representation.

3.2 What is a Spinor?

The simplest spinor representations are $d_L = (\frac{1}{2}, 0)$ and $d_R = (0, \frac{1}{2})$. We call these the left and right-handed Weyl spinor representations, the reason we call them left and right will be explained next lecture. One transforms as a Pauli spinor under J_i^+ but as a scalar (trivially) under J_i^-, while the other does the opposite. That is,

$$d_L(J_i^+) = \tfrac{1}{2}\sigma, \quad d_L(J_i^-) = 0, \qquad d_L(J_i) = \frac{1}{2}\sigma_i, \quad d_L(K) = -\frac{i}{2}\sigma_i$$

Or, written in another way, the matrix representation of each generator is as follows.

$$d_L(\mathcal{J}^{\mu\nu}) = \frac{1}{2}\begin{pmatrix} 0 & -i\sigma_1 & -i\sigma_2 & -i\sigma_3 \\ i\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ i\sigma_2 & -\sigma_3 & 0 & \sigma_1 \\ i\sigma_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

Both d_L and d_R are 2-dimensional representations. One describes left-handed spin-1/2 particles, while the other describes right-handed spin-1/2 particles. If we want particles that are not chiral, and also have a mass term, we can take their direct sum, which is called the Dirac representation.

$$s = d_L \oplus d_R = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})$$

Note that it is not irreducible, and has dimension $2 + 2 = 4$. That is, Dirac spinors are 4 component fields. Using the matrices above, it is straightforwards to show that for the Dirac representation,

$$d(\mathcal{J}^{\mu\nu}) = -\frac{i}{4}[\sigma^\mu, \sigma^\nu] = \Sigma^{\mu\nu}$$

which you should be familiar with from lecture. This consistency check shows that the bispinor representation describes the Dirac particles discussed in class. To obtain the irreps of the Lorentz group, we just exponentiate the irreps of the Lorentz algebra. For instance, for the Dirac representation we have:

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu} d(\mathcal{J}^{\mu\nu})} = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

To conclude, we list out the low dimensional representations of the Lorentz group.

Representation	(j_+, j_-)	Dimension	Irrep?	(Beyond) SM Fields
Scalar	$(0,0)$	1	Y	Higgs, (Dilaton), (Axion)
Left Weyl	$(\frac{1}{2},0)$	2	Y	Left-handed Neutrinos
Right Weyl	$(0,\frac{1}{2})$	2	Y	(Right-handed Neutrinos)
Dirac/bispinor	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	4	N	Electrons, Quarks
Vector	$(\frac{1}{2},\frac{1}{2})$	4	N	Photon, Gluon, W, Z
Antisymmetric Tensor	$(1,0)\oplus(0,1)$	6	N	(Kalb-Ramond)
Rarita-Schwinger	$(1,\frac{1}{2})$, $(\frac{1}{2},1)$	6	Y	(Gravitino)
Traceless Symmetric Tensor	$(1,1)$	9	Y	(Graviton)

Recitation 8

Wentao Cui

April 14, 2023

1 Vector and Axial Symmetries

1.1 Chiralty

We work in the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4$, and satisfies

$$(\gamma_5)^\dagger = \gamma_5, \qquad \{\gamma^\mu, \gamma_5\} = 0$$

This representation is useful for the purpose of chirality, because γ^5 is diagonal. The $+1$ eigenspace corresponds to left-handed spinors ψ_L, spanned by Dirac spinors non-zero in the top 2 components. Similarly, the -1 eigenspace corresponds to right-handed spinors ψ_R, spanned by Dirac spinors non-zero in the bottom 2 components. Therefore, we a Dirac spinor can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Note that this is consistent with last recitation, where we have written the Dirac representation as the direct sum of left and right-handed Weyl representations:

$$\text{Dirac} = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) = \text{Weyl}_L + \text{Weyl}_R$$

An arbitrary Dirac spinor can be projected onto its left/right-handed subspaces via projection operators:

$$P_L = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Because these are projectors, they satisfy:

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0, \qquad P_L + P_R = 1$$

Slightly abusing notation by writing $\psi_{L/R} = P_{L/R}\psi$, some useful identities are:

$$P_L\gamma^\mu = \gamma^\mu P_R, \qquad P_R\gamma^\mu = \gamma^\mu P_L, \qquad \bar{\psi}_L = \bar{\psi}P_R, \qquad \bar{\psi}_R = \bar{\psi}P_L$$

These are very important in the Standard Model because it is a chiral theory: the particles mediating forces couple differently to left and right-handed fermions.

1.2 Vector and Axial Transformations

Now consider the Lagrangian for a Dirac spinor,

$$\mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi$$

This is invariant under the vector transformation $\psi(x) \to e^{i\alpha}\psi(x)$. The conserved quantity is the vector current, $j_V^\mu = \bar{\psi}\gamma^\mu\psi$. From the decomposition above, we see that this rotates left and right-handed spinors in the same way:

$$\psi_L \to e^{i\alpha}\psi_L, \qquad \psi_R \to e^{i\alpha}\psi_R$$

We may also consider the axial transformation $\psi(x) \to e^{i\alpha\gamma_5}\psi(x)$. This rotates left and right-handed spinors in opposite ways:

$$\psi_L \to e^{i\alpha}\psi_L, \qquad \psi_R \to e^{-i\alpha}\psi_R$$

Note that the a Dirac mass term $m\bar{\psi}\psi$ is not invariant under this transformation:

$$m\bar{\psi}\psi \to m\psi^\dagger e^{-i\alpha\gamma_5}\gamma^0 e^{i\alpha\gamma_5}\psi = m\psi^\dagger\gamma^0 e^{2i\alpha(\gamma_5)^2}\psi = e^{2i\alpha}m\bar{\psi}\psi$$

Meanwhile, the kinetic term is invariant:

$$-i\bar{\psi}\partial\!\!\!/\psi \to -i\psi^\dagger e^{-i\alpha\gamma_5}\gamma^0\partial\!\!\!/ e^{i\alpha\gamma_5}\psi = -i\bar{\psi}e^{i\alpha\gamma_5}\partial\!\!\!/ e^{i\alpha\gamma_5}\psi = -i\bar{\psi}\partial\!\!\!/ e^{-i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = -i\bar{\psi}\partial\!\!\!/\psi$$

Therefore, this is a symmetry of $\mathcal{L}$ only for a massless Dirac spinor $m = 0$. The conserved quantity is the vector current, $j_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$.

To shed more light on this, it is instructive to write the $\mathcal{L}$ in terms of ψ_L and ψ_R:

$$\bar{\psi}\partial\!\!\!/\psi = \bar{\psi}\partial\!\!\!/(P_L + P_R)\psi = \bar{\psi}\partial\!\!\!/P_L P_L\psi + \bar{\psi}\partial\!\!\!/P_R P_R\psi = \bar{\psi}P_R\partial\!\!\!/P_L\psi + \bar{\psi}P_L\partial\!\!\!/P_R\psi = \bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R$$
$$m\bar{\psi}\psi = \bar{\psi}(P_L + P_R)\psi = \bar{\psi}P_L P_L\psi + \bar{\psi}P_R P_R\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R$$

The kinetic term decomposes into a kinetic term for ψ_L and a kinetic term for ψ_R. The mass term couples the left and right-handed Weyl spinors. When $m = 0$ the theory becomes decoupled, and the Dirac Lagrangian reduces to a free ψ_L and free ψ_R.

How now, do the vector and axial symmeties manifest? The massless Lagrangian $\mathcal{L} = -i(\bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R)$ has the symmetries $\psi_L \to e^{i\alpha_L}\psi_L$, and $\psi_R \to e^{i\alpha_L}\psi_R$, where we rotate the left spinor by α_L, and the right by α_R independently. Equivalently, we may rotate left/right by the same $e^{i\alpha_V}$, or by opposites $e^{\pm\alpha_A}$. These are the vector and axial symmetries. The coupling term $m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ is not invariant if we rotate $\psi_{L,R}$ differently.

1.3 The Chiral Anomaly

The moral of the previous section is that the Lagrangian for a massless Dirac fermion enjoys the vector and axial symmetries. Noethers theorem tells us that the corresponding currents are conserved:

$$\partial_\mu j_V^\mu = \partial_\mu j_A^\mu = 0$$

However, this symmetry is broken when we quantize the theory: if we couple a massless Dirac fermion to an EM field, we find that the Noether current is not conserved:

$$\partial_\mu j_A^0 = \frac{\alpha}{4\pi}e^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{\alpha}{4\pi}F \wedge F$$

Here $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, which may be familiar to you from your undergraduate EM class.

Generally, an anomaly is a symmetry of a classical theory, which is not a symmetry of the corresponding quantum theory. The one we have just seen is called the chiral or Adler-Bell-Jackiw anomaly. It is responsible for much physics, such as the very short lifetime of the neutral pion, by mediating the would-be-forbidden process $\pi^0 \to \gamma\gamma$.

Where does Noether's theorem break down? The anomaly is easiest to see using the path-integral, which is the central object in a quantum theory. We can write

$$Z = \int D\psi D\bar{\psi} e^{iS[\bar{\psi},\psi]} = \int D\psi' D\bar{\psi}' e^{iS[\bar{\psi}',\psi']}$$

Given a symmetry of the action $(\psi, \bar{\psi}) \to (\psi', \bar{\psi}')$, the Lagrangian density must change by only a total derivative (i.e. a surface term):

$$iS[\psi', \bar{\psi}'] = iS[\psi, \bar{\psi}] + \int d^4x \partial_\mu j_A^\mu(x)$$

Noether's procedure shows how to construct j_A^μ such that $\partial_\mu j_A^\mu(x) = 0$. However, this is predicated on the assumption that the path-integral measure is invariant under our symmetry, $D\psi' D\bar{\psi}' = D\psi D\bar{\psi}$. In general this is not true. Instead,

$$D\psi' D\bar{\psi}' = D\psi D\bar{\psi} \det \Delta^{-1} = D\psi D\bar{\psi} e^{\ln \det \Delta^{-1}} = D\psi D\bar{\psi} e^{\mathrm{Tr} \ln \Delta^{-1}} = D\psi D\bar{\psi} e^{-\int d^4x \ln \Delta}$$

Putting everything together, we have

$$\begin{aligned}
\int D\psi D\bar{\psi} e^{iS[\bar{\psi},\psi]} &= \int D\psi' D\bar{\psi}' e^{iS[\bar{\psi}',\psi']} \\
&= \int D\psi D\bar{\psi} e^{-\int d^4x \ln \Delta} e^{iS[\psi,\bar{\psi}] + \int d^4x \partial_\mu j_A^\mu(x)} \\
&= \int D\psi D\bar{\psi} e^{iS[\psi,\bar{\psi}]} e^{\int d^4x (\partial_\mu j_A^\mu(x) - \ln \Delta)}
\end{aligned}$$

For these to be equal, we must have that

$$\partial_\mu j_A^\mu = \ln \Delta$$

That is, the current j_A^μ is no longer conserved.

2 Weyl, Dirac, Majorana

Moved to next time.

Recitation 10

Wentao Cui

April 28, 2023

1 The Electroweak Theory and the Higgs Mechanism

1.1 A Toy Model

We now have enough background to discuss one of the most interesting aspects of the Standard Model. In particular, we'll be focusing on the electroweak sector.

The electroweak sector of the Standard Model is a chiral gauge theory, described by the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_L$. Don't worry about the word 'gauge', the takeaway is that the electroweak Lagrangian is invariant under local $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ transformations. The subscript L here is important, because it means that the SU(2) transformation only acts on left-handed particles. This is why the Standard Model is a chiral theory–it treats left-hand particles differently from right-handed ones. The U(1) is called hypercharge, which is related to but different from $\mathrm{U}(1)_{\mathrm{EM}}$.

Toy Model: Symmetry Structure
Let's make this concrete with a toy model with 2 global symmetries $\mathrm{U}(1)_L \times \mathrm{U}(1)_Y$. Instead of $\mathrm{SU}(2)_L$ we'll consider $\mathrm{U}(1)_L$, which rotates left and right-handed Weyl fields as

$$\psi_L \to e^{i\alpha Q}\psi_L, \qquad \psi_R \to \psi_R$$

We also have a hypercharge symmetry, which rotates left and right-handed fermions equally:

$$\psi_L \to e^{i\beta Y}\psi_L, \qquad \psi_R \to e^{i\beta Y}\psi_R$$

This one we've seen, but the chiral $\mathrm{U}(1)_L$ is new. Note that here Q and Y are charge operators: let's assign different charges to different particles. The particles we will consider are electrons and neutrinos (there is no evidence of right-handed neutrinos, but more on this later)

What does this have to do with the Standard Model? After all, the spinors around us, like electrons and positrons, are Dirac spinors. This is not so in the Standard Model: the fermionic matter is described by Weyl spinors. To see why, consider a Dirac spinor charged under $\mathrm{U}(1)_L$ with $Q = 1$, $\Psi = (\psi_L, \psi_R)^\dagger$. The problem is that the mass term is not invariant under the $\mathrm{U}(1)_L$ symmetry:

$$\bar{\Psi}\Psi = \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \to e^{i\alpha}\psi_R^\dagger \psi_L + e^{-i\alpha}\psi_L^\dagger \psi_R$$

This shouldn't be surprising, since the mass term couples the left and right Weyl fermions, which transform differently. Therefore, the Lagrangian has 2 independent Weyl fermions $\psi_{L,R}$.

Toy Model: Matter Structure
So what are our fields? We need to work with Weyl fermions. Scalars, vectors, and so on can be added, but there is no point in adding Dirac spinors, since massless ones are equivalent to Weyl spinors. Since ψ_L and ψ_R are decoupled, they should be thought of as distinct particles, and for instance we can have

more of one-handed particles than we do the other. Inspired by the real world, the fermion fields we'll work with are a LH electron, RH electron, and LH neutrino:

$$e_L \qquad e_R \qquad \nu_L$$

I stress that e_L and e_R are decoupled fields: so far, there is no reason to treat them as chiral components of the same field. Now that we have our fields, we need to specify their charges under each symmetry. We do this in a table:

Field	$\mathrm{U}(1)_L$	$\mathrm{U}(1)_Y$	Lorentz
e_L	$-1/2$	$-1/2$	$(1/2, 0)$
e_R	0	-1	$(0, 1/2)$
ν_L	$1/2$	$-1/2$	$(1/2, 0)$

The $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_Y$ charges are taken from the Standard Model, and may seem arbitrary to you. However, in our world they are almost uniquely constrained by anomaly cancellation conditions.

We can now write down mass terms. Any mass term in our Lagrangian must be Lorentz-invariant, which constrain us to Majorana and Dirac terms. To add them to the Lagrangian, we must also check they transform trivially under both $U(1)_L$ and $U(1)_Y$.

- Majorana masses: $e_L^T \sigma^2 e_L, \qquad e_R^T \sigma^2 e_R, \qquad \nu_L^T \sigma^2 \nu_L, \qquad \nu_L^T \sigma^2 e_L, \qquad e_L^T \sigma^2 \nu_L$
  These are not invariant under either the $\mathrm{U}(1)_L$ or $\mathrm{U}(1)_Y$ symmetries, so are not valid terms in $\mathcal{L}$.
- Dirac masses: $e_L^\dagger e_R + \text{h.c.}, \qquad \nu_L^\dagger e_R + \text{h.c.}$
  These are not invariant under $U(1)_Y$.

The takeaway from this is that any mass term we can construct is not invariant under our toy electroweak symmetry $\mathrm{U}(1)_L \times \mathrm{U}(1)_Y$. But real world electrons and neutrinos have masses, and the 2 components $e_{L,R}$ can be thought of as a Dirac fermion with a Dirac mass. We must account for this somehow if our toy theory (and the Standard Model) is to hold any water.

1.2 The Higgs Mechanism

Enter the Higgs boson. The Higgs is a scalar H whose existence allows e_L and e_R to couple together. In the electroweak sector, this is done via a Yukawa coupling,

$$\mathcal{L}_{\text{EW}} \supset \lambda (H^\dagger e_R^\dagger e_L + \text{h.c.})$$

Since this is a term in our Lagrangian, it must be invariant under $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_Y$. We are therefore able to deduce the U(1) charges of the Higgs.

Field	$\mathrm{U}(1)_L$	$\mathrm{U}(1)_Y$	Lorentz
H	$-1/2$	$1/2$	$(0, 0)$

The details of the Higgs mechanism are quite intricate, but here I'll provide a basic description. Through a process called spontaneous symmetry breaking (explain if have time: Mexican hat potential), below the energy of $v \sim 246\text{GeV}$, the Higgs boson takes on a 'vacuum expectation value' (vev), and can be parameterized as:

$$H(x) = \frac{1}{\sqrt{2}}(v + h(x))$$

v is this constant energy scale, and $h(x)$ is the physical Higgs boson, which is a real scalar field. It parameterizes fluctuations about its classical value v. Note that this parameterization is only valid for energies $E \ll v$: for larger energies, $h(x)$ is no longer a perturbation and this breaks down. However, 246GeV is an absurdly high energy: the LHC can only probe up to 14TeV. It is of the same order as the Higgs mass $m_h = v/2 \sim 125\text{GeV}$, and this is why it took so long to find the Higgs.

Let's take this parameterization, and substitute it into our coupling terms in our Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{EW}}|_{E\ll v} &\supset \frac{\lambda}{\sqrt{2}}[(v+h)^\dagger e_R^\dagger e_L + e_L^\dagger e_R(v+h)] \\ &= \frac{\lambda v}{\sqrt{2}}(e_R^\dagger e_L + e_L^\dagger e_R) + \frac{\lambda}{2}(h^\dagger e_R^\dagger e_L + e_L^\dagger e_R h) \\ &= \frac{\lambda v}{\sqrt{2}}\bar{e}e + \frac{\lambda}{\sqrt{2}}h\bar{e}e\end{aligned}$$

In the second line, we have generated a Dirac mass out of $e_{L,R}$. In the last line, we have combined e_L and e_R into a Dirac spinor field, $e = (e_L, e_R)^T$. We have also generatde 3-point interactions between the electron and the Higgs. (Draw Feynman rules before and after SSB.)

The takeaway from this is that physics splits into 2 regimes:

- $E \gtrsim v$ (very high energy): SSB has not occered, and left/right handed e_L, e_R are massless, independent particles which couple to the Higgs via a 3-point interaction.
- $E \ll v$ ('low' energy): the Higgs has given the electrons a Dirac mass by taking on a vev., in a way that respects chiral gauge symmetry.

The Higgs mechanism in the Standard Model is slightly more complicated, owing to that the chiral symmetry is $\text{SU}(2)_L$ instead of our toy $\text{U}(1)_L$. It also allows for a host of other processes, including the generation of a mass for gauge bosons, and the generation of a photon as a linear combination of the $\text{U}(1)_L$ and $U(1)_Y$ symmetries.

Neutrino Masses
Note that the Higgs cannot generate a neutrino mass not happen for neutrinos, because of the absence of a right-handed ν_R. This coincides with our world: all neutrinos we have observed are left-handed, and we do not know why. Similarly, the Standard Model does not predict neutrino masses. On the other hand, we know neutrino masses exist via observing neutrino oscillations. This discrepency can be accounted for by adding neutrino masses *to* the Standard Model. There are two easy ways to do this, but the form of the masses depends on whether neutrinos are Dirac or Majorana, something else we do not know.

- If they are Dirac, then there must be a right-handed neutrino ν_R which we have not observed. If it exists we expect it to be very heavy. Neutrino masses can thus be generated by the Higgs, as

$$\mathcal{L}_{\text{SM}} \supset \lambda(H^\dagger \nu_R^\dagger \nu_L + \text{h.c.})$$

- If they are Majorana, there does not need to be a right-handed ν_R. We can write the mass term

$$\mathcal{L}_{\text{SM}} \supset \lambda(\nu_L \sigma_2 H)(\nu_L \sigma_2 H)$$

  From dimensional reasons, we expect the coefficient λ to by incredibly small, which explains why neutrinos are so light. But even if this is the case, we still need to explain the absence of the right-handed ν_R.

All in all, neutrinos are one of the least understood aspects of the Standard Model, and a fertile source to look for new physics.